

SECOND-ORDER CONSTRAINED VARIATIONAL PROBLEMS ON LIE ALGEBROIDS: APPLICATIONS TO OPTIMAL CONTROL

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ABSTRACT. The aim of this work is to study, from an intrinsic and geometric point of view, second-order constrained variational problems on Lie algebroids, that is, optimization problems defined by a cost function which depends on higher-order derivatives of admissible curves on a Lie algebroid. Extending the classical Skinner and Rusk formalism for the mechanics in the context of Lie algebroids, for second-order constrained mechanical systems, we derive the corresponding dynamical equations. We find a symplectic Lie subalgebroid where, under some mild regularity conditions, the second-order constrained variational problem, seen as a presymplectic Hamiltonian system, has a unique solution. We study the relationship of this formalism with the second-order constrained Euler-Poincaré and Lagrange-Poincaré equations, among others. Our study is applied to the optimal control of mechanical systems.

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1. INTRODUCTION

Lie algebroids have deserved a lot of interest in recent years. Since a Lie algebroid is a concept which unifies tangent bundles and Lie algebras, one can suspect their relation

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with mechanics. More precisely, a Lie algebroid over a manifold Q is a vector bundle $\tau_E : E \rightarrow Q$ over Q with a Lie algebra structure over the space $\Gamma(\tau_E)$ of sections of E and an application $\rho : E \rightarrow TQ$ called anchor map satisfying some compatibility conditions (see [51]). Examples of Lie algebroids are the tangent bundle over a manifold Q where the Lie bracket is the usual Lie bracket of vector fields and the anchor map is the identity function; a real finite dimensional Lie algebras as vector bundles over a point, where the anchor map is the null application; action Lie algebroids of the type $pr_1 : M \times \mathfrak{g} \rightarrow M$ where \mathfrak{g} is a Lie algebra acting infinitesimally over the manifold M with a Lie bracket over the space of sections induced by the Lie algebra structure and whose anchor map is the action of \mathfrak{g} over M ; and, the Lie-Atiyah algebroid $\tau_{TQ/G} : TQ/G \rightarrow \widehat{M} = Q/G$ associated with the G -principal bundle $p : Q \rightarrow \widehat{M}$ where the anchor map is induced by the tangent application of p , $Tp : TQ \rightarrow T\widehat{M}$ [49, 51, 57, 71].

In [71] Alan Weinstein developed a generalized theory of Lagrangian mechanics on Lie algebroids and he obtained the equations of motion using the linear Poisson structure on the dual of the Lie algebroid and the Legendre transformation associated with a regular Lagrangian $L : E \rightarrow \mathbb{R}$. In [71] also he asked about whether it is possible to develop a formalism similar on Lie algebroids to Klein's formalism [46] in Lagrangian mechanics. This task was obtained by Eduardo Martínez in [57] (see also [56]). The main notion is that of prolongation of a Lie algebroid over a mapping introduced by Higgins and Mackenzie in [51]. A more general situation, the prolongation of an anchored bundle $\tau_E : E \rightarrow Q$ was also considered by Popescu in [65, 66].

The importance of Lie algebroids in mathematics is beyond doubt and in the last years Lie algebroids has been a lot of applications in theoretical physics and other related sciences. More concretely in Classical Mechanics, Classical Field Theory and their applications. One of the main characteristic concerning that Lie algebroids are interesting in Classical Mechanics lie in the fact that there are many different situations that can be understand in a general framework using the theory of Lie algebroids as systems with symmetries, systems over semidirect products, Hamiltonian and Lagrangian systems, systems with constraints (nonholonomic and vakonomic) and Classical Fields theory [1, 14, 15, 10, 16, 25, 26, 32, 47, 52, 61].

In [49] M. de León, J.C Marrero and E. Martínez have developed a Hamiltonian description for the mechanics on Lie algebroids and they have shown that the dynamics is obtained solving an equation in the same way than in Classical Mechanics (see also [56] and [71]). Moreover, they shown that the Legendre transformation $leg_L : E \rightarrow E^*$ associated to the Lagrangian $L : E \rightarrow \mathbb{R}$ induces a Lie algebroid morphism and when the Lagrangian is regular both formalisms are equivalent.

Marrero and collaborators also have analyzed the case of non-holonomic mechanics on Lie algebroids [25]. In other direction, in [40] D. Iglesias, J.C. Marrero, D. Martín de Diego and D. Sosa have studied singular Lagrangian systems and vakonomic mechanics from the point of view of Lie algebroids obtained through the application of a constrained variational principle. They have developed a constraint algorithm for presymplectic Lie algebroids generalizing the well know constraint algorithm of Gotay, Nester and Hinds [36, 37] and they also have established the Skinner and Rusk formalism on Lie algebroids. Some of the results given are as an extension of this framework for constrained second-order systems.

Our framework is based in the Skinner-Rusk formalism which combines simultaneously some features of the Lagrangian and Hamiltonian classical formalisms. The idea of this formulation was to obtain a common framework for both regular and singular dynamics, obtaining simultaneously the Hamiltonian and Lagrangian formulations of the dynamics. Over the years, however, Skinner and Rusk's framework was extended in many directions: It was originally developed for first-order autonomous mechanical systems [70], and later generalized to non-autonomous dynamical systems [2, 24, 68], control systems [4] and, more recently to classical field theories [12, 28].

Briefly, in this formulation, one starts with a differentiable manifold Q as the configuration space, and the Whitney sum $TQ \oplus T^*Q$ as the evolution space (with canonical projections $\pi_1 : TQ \oplus T^*Q \rightarrow TQ$ and $\pi_2 : TQ \oplus T^*Q \rightarrow T^*Q$). Define on $TQ \oplus T^*Q$ the presymplectic 2-form $\Omega = \pi_2^* \omega_Q$, where ω_Q is the canonical symplectic form on T^*Q , and observe that the rank of this presymplectic form is everywhere equal to $2n$. If the dynamical system under consideration admits a Lagrangian description, with Lagrangian $L \in C^\infty(TQ)$, then one can obtain a (presymplectic)-Hamiltonian representation on $TQ \oplus T^*Q$ given by the presymplectic 2-form Ω and the Hamiltonian function $H = \langle \pi_1, \pi_2 \rangle - \pi_1^* L$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors on Q . In this Hamiltonian system the dynamics is given by vector fields X , which are solutions to the Hamiltonian equation $i_X \Omega = dH$. If L is regular, then there exists a unique vector field X solution to the previous equation, which is tangent to the graph of the Legendre map. In the singular case, it is necessary to develop a constraint algorithm in order to find a submanifold (if it exists) where there exists a well-defined dynamical vector field.

Recently, higher-order variational problems have been studied for their important applications in aeronautics, robotics, computer-aided design, air traffic control, trajectory planning, and in general, problems of interpolation and approximation of curves on Riemannian manifolds [6, 11, 39, 45, 50, 62, 60, 63]. There are variational principles which involves higher-order derivatives by Gay Balmaz et.al., [29, 30, 31], (see also [48]) since from it one can obtain the equations of motion for Lagrangians where the configuration space is a higher-order tangent bundle. More recently, there have been an interest in study of the geometrical structures associated with higher order variational problems with the aim of a deepest understanding of those geometric structures [20, 23, 67, 58, 42, 43, 44] as well the relation of higher-order mechanics and graded bundles, [8, 9, 10].

In this work, we study a geometric framework, based on the Skinner and Rusk formalism, for constrained second-order variational problems determined by a Lagrangian function, playing the role of cost function in an optimal control problem, which depends on derivatives of admissible curves on a Lie algebroid. The strategy is to apply the geometric procedure described above in combination with an extension of the constraint algorithm developed by Gotay, Nester and Hinds [36, 37] in the setting of Lie algebroids [40]. Our work permits to obtain constrained second-order Euler-Lagrange equations, Euler-Poincaré, Lagrange-Poincaré equations in an unified framework and understand the geometric structures subjacent in second-order variational problems. We show how this study can be applied to the problem of finding necessary conditions for optimality in optimal control problems of mechanical system with symmetries, where trajectories are parameterized by the admissible controls and the necessary conditions for extremals in the optimal control problem are expressed using a pseudo-Hamiltonian formulation based on the Pontryagin maximum principle.

The paper is organized as follows. In Section 2 we introduce some known notions concerning Lie algebroids that are necessary for further developments in this work. In section 3 we will use the notion of Lie algebroid and prolongation of a Lie algebroid described in 2 to derive the Euler-Lagrange equations and Hamilton equations on Lie algebroids. Next, after introduce the constraint algorithm for presymplectic Lie algebroids and study vakonomic mechanics on Lie algebroids, we study the geometric formalism for second-order constrained variational problems using and adaptation of the classical Skinner-Rusk formalism for the second-order constrained systems on Lie algebroids. In section 4 we study optimal control problems of mechanical systems defined on Lie algebroids. Optimality conditions for the optimal control of the Elroy's Beanie are derived. Several examples show how to apply the techniques along all the work.

2. LIE ALGEBROIDS AND ADMISSIBLE ELEMENTS

In this section, we introduce some known notions and develop new concepts concerning Lie algebroids that are necessary for further developments in this work. We illustrate the theory with several examples. We refer the reader to [13, 51] for more details about Lie algebroids and their role in differential geometry.

2.1. Lie algebroids, Lie subalgebroids and Cartan calculus on Lie algebroids.

Definition 2.1. Let E be a vector bundle of rank n over a manifold M of dimension m . A *Lie algebroid structure* on the vector bundle $\tau_E : E \rightarrow M$ is a \mathbb{R} -linear bracket $[\cdot, \cdot] : \Gamma(\tau_E) \times \Gamma(\tau_E) \rightarrow \Gamma(\tau_E)$ on the space $\Gamma(\tau_E)$, the $C^\infty(M)$ -module of sections of E , and a vector bundle morphism $\rho : E \rightarrow TM$, the *anchor map*, such that:

- (1) The bracket $[\cdot, \cdot]$ satisfies the Jacobi identity, that is,

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \quad \forall X, Y, Z \in \Gamma(\tau_E).$$

- (2) If we also denote by $\rho : \Gamma(\tau_E) \rightarrow \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$ -modules induced by the anchor map then

$$[X, fY] = f[X, Y] + \rho(X)(f)Y, \quad \text{for } X, Y \in \Gamma(\tau_E) \text{ and } f \in C^\infty(M). \quad (1)$$

We will said that the triple $(E, [\cdot, \cdot], \rho)$ is a *Lie algebroid* over M . In this context, sections of τ_E , play the role of vector fields on M , and the sections of the dual bundle $\tau_E^* : E^* \rightarrow M$ of 1-forms on M .

We may consider two type of distinguished functions: given $f \in C^\infty(M)$ one may define a function \tilde{f} on E by $\tilde{f} = f \circ \tau_E$, the *basic functions*. And, given a section θ of the dual bundle $\tau_E^* : E^* \rightarrow M$, may be regarded as a *lineal function* $\hat{\theta}$ on E as $\hat{\theta}(e) = \langle \theta(\tau_E(e)), e \rangle$ for all $e \in E$. In this sense, $\Gamma(\tau_E)$ is locally generated by the differential of basic and linear functions.

If $X, Y, Z \in \Gamma(\tau_E)$ and $f \in C^\infty(M)$, then using the Jacobi identity we obtain that

$$[[X, Y], fZ] = f[[X, [Y, Z]]] + [\rho(X), \rho(Y)](f)Z. \quad (2)$$

Also, from (1) it follows that

$$[[X, Y], fZ] = f[[[X, Y], Z] + \rho[X, Y](f)Z]. \quad (3)$$

Then, using (2) and (3) and the fact that $[\cdot, \cdot]$ is a Lie bracket we conclude that

$$\rho[X, Y] = [\rho(X), \rho(Y)],$$

that is, $\rho : \Gamma(\tau_E) \rightarrow \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(\tau_E), [\cdot, \cdot])$ and $(\mathfrak{X}, [\cdot, \cdot])$.

The algebra $\bigoplus_k \Gamma(\Lambda^k E^*)$ of multisections of τ_E^* plays the role of the algebra of the differential forms and it is possible to define a *differential operator* as follow:

Definition 2.2. If $(E, [\cdot, \cdot], \rho)$ is a Lie algebroid over M , one can be define the *differential* of E , $d^E : \Gamma(\bigwedge^k \tau_E^*) \rightarrow \Gamma(\bigwedge^{k+1} \tau_E^*)$, as follows;

$$\begin{aligned} d^E \mu(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i \rho(X_i)(\mu(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \sum_{i < j} (-1)^{i+j} \mu([X, Y], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k), \end{aligned}$$

for $\mu \in \Gamma(\bigwedge^k \tau_E^*)$ and $X_0, \dots, X_k \in \Gamma(\tau_E)$.

From the properties of Lie algebroids it follows that d^E is a cohomology operator, that is, $(d^E)^2 = 0$ and $d^E(\alpha \wedge \beta) = d^E \alpha \wedge \beta + (-1)^k \alpha \wedge d^E \beta$, for $\alpha \in \Gamma(\Lambda^k E^*)$ and $\beta \in \Gamma(\Lambda^r E^*)$ (see [51] for more details).

Conversely it is possible to recover the Lie algebroid structure of E from the existence of an exterior differential on $\Gamma(\Lambda^\bullet \tau_E^*)$. If $f : M \rightarrow \mathbb{R}$ is a real smooth function, one can define the anchor map and the Lie bracket as follows:

- (1) $d^E f(X) = \rho(X)f$, for $X \in \Gamma(\tau_E)$,
- (2) $i_{\llbracket X, Y \rrbracket} \theta = \rho(X)\theta(Y) - \rho(Y)\theta(X) - d^E \theta(X, Y)$ for all $X, Y \in \Gamma(\tau_E)$ and $\theta \in \Gamma(\tau_{E^*})$.

Moreover, from the last equality, the section $\theta \in \Gamma(\tau_{E^*})$ is a *1-cocycle* if and only if $d^E \theta = 0$, or, equivalently,

$$\theta[\llbracket X, Y \rrbracket] = \rho(X)(\theta(Y)) - \rho(Y)(\theta(X)),$$

for all $X, Y \in \Gamma(\tau_E)$.

We may also define the *Lie derivative* with respect to a section $X \in \Gamma(\tau_E)$ as the operator $\mathcal{L}_X^E : \Gamma(\bigwedge^k \tau_{E^*}) \rightarrow \Gamma(\bigwedge^k \tau_{E^*})$ given by

$$\mathcal{L}_X^E \theta = i_X \circ d^E \theta + d^E \circ i_X \theta,$$

for $\theta \in \Gamma(\bigwedge^k \tau_{E^*})$. One also has the usual identities

- (1) $d^E \circ \mathcal{L}_X^E = \mathcal{L}_X^E \circ d^E$,
- (2) $\mathcal{L}_X^E i_Y - i_X \mathcal{L}_Y^E = i_{\llbracket X, Y \rrbracket}$,
- (3) $\mathcal{L}_X^E \mathcal{L}_Y^E - \mathcal{L}_Y^E \mathcal{L}_X^E = \mathcal{L}_{\llbracket X, Y \rrbracket}^E$.

We take local coordinates (x^i) on M with $i = 1, \dots, m$ and a local basis $\{e_A\}$ of sections of the vector bundle $\tau_E : E \rightarrow M$ with $A = 1, \dots, n$, then we have the corresponding local coordinates on an open subset $\tau_E^{-1}(U)$ of E , (x^i, y^A) (U is an open subset of Q), where $y^A(e)$ is the A -th coordinate of $e \in E$ in the given basis i.e., every $e \in E$ is expressed as $e = y^1 e_1(\tau_E(e)) + \dots + y^n e_n(\tau_E(e))$.

Such coordinates determine the local functions ρ_A^i, C_{AB}^C on M which contain the local information of the Lie algebroid structure, and accordingly they are called *structure functions of the Lie algebroid*. These are given by

$$\rho(e_A) = \rho_A^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \llbracket e_A, e_B \rrbracket = C_{AB}^C e_C. \quad (4)$$

These functions should satisfy the relations

$$\rho_A^j \frac{\partial \rho_B^i}{\partial x^j} - \rho_B^j \frac{\partial \rho_A^i}{\partial x^j} = \rho_C^i C_{AB}^C \quad (5)$$

and

$$\sum_{cyclic(A,B,C)} \left[\rho_A^i \frac{\partial C_{BC}^D}{\partial x^i} + C_{AF}^D C_{BC}^F \right] = 0, \quad (6)$$

which are usually called *the structure equations*.

If $f \in C^\infty(M)$,

$$d^E f = \frac{\partial f}{\partial x^i} \rho_A^i e^A, \quad (7)$$

where $\{e^A\}$ is the dual basis of $\{e_A\}$. If $\theta \in \Gamma(\tau_{E^*})$ and $\theta = \theta_C e^C$ it follows that

$$d^E \theta = \left(\frac{\partial \theta_C}{\partial x^i} \rho_B^i - \frac{1}{2} \theta_A C_{BC}^A \right) e^B \wedge e^C. \quad (8)$$

In particular,

$$d^E x^i = \rho_A^i e^A, \quad d^E e^A = -\frac{1}{2} C_{BC}^A e^B \wedge e^C.$$

2.1.1. Examples of Lie algebroids.

Example 1. Given a *finite dimensional real Lie algebra* \mathfrak{g} and $M = \{m\}$ be a unique point, we consider the vector bundle $\tau_{\mathfrak{g}} : \mathfrak{g} \rightarrow M$. The sections of this bundle can be identified with the elements of \mathfrak{g} and therefore we can consider as the Lie bracket the structure of the Lie algebra induced by \mathfrak{g} , and denoted by $[\cdot, \cdot]_{\mathfrak{g}}$. Since $TM = \{0\}$ one may consider the anchor map $\rho \equiv 0$. The triple $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, 0)$ is a Lie algebroid over a point.

Example 2. Consider a *tangent bundle* of a manifold M . The sections of the bundle $\tau_{TM} : TM \rightarrow M$ are the set of vector fields on M . The anchor map $\rho : TM \rightarrow TM$ is the identity function and the Lie bracket defined on $\Gamma(\tau_{TM})$ is induced by the Lie bracket of vector fields on M .

Example 3. Let $\phi : M \times G \rightarrow M$ be an action of G on the manifold M where G is a Lie group. The induced anti-homomorphism between the Lie algebras \mathfrak{g} and $\mathfrak{X}(M)$ by the action is determined by $\Phi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$, $\xi \mapsto \xi_M$, where ξ_M is the infinitesimal generator of the action for $\xi \in \mathfrak{g}$.

The vector bundle $\tau_{M \times \mathfrak{g}} : M \times \mathfrak{g} \rightarrow M$ is a Lie algebroid over M . The anchor map $\rho : M \times \mathfrak{g} \rightarrow TM$, is defined by $\rho(m, \xi) = -\xi_M(m)$ and the Lie bracket of sections is given by the Lie algebra structure on $\Gamma(\tau_{M \times \mathfrak{g}})$ as

$$[\hat{\xi}, \hat{\eta}]_{M \times \mathfrak{g}}(m) = (m, [\xi, \eta]) = \widehat{[\xi, \eta]}$$

for $m \in M$, where $\hat{\xi}(m) = (m, \xi)$, $\hat{\eta}(m) = (m, \eta)$ for $\xi, \eta \in \mathfrak{g}$. The triple $(M \times \mathfrak{g}, \rho, [\cdot, \cdot]_{M \times \mathfrak{g}})$ is called *Action Lie algebroid*.

Example 4. Let G be a Lie group and we assume that G acts free and properly on M . We denote by $\pi : M \rightarrow \widehat{M} = M/G$ the associated principal bundle. The tangent lift of the action gives a free and proper action of G on TM and $\widehat{TM} = TM/G$ is a quotient manifold. The quotient vector bundle $\tau_{\widehat{TM}} : \widehat{TM} \rightarrow \widehat{M}$ where $\tau_{\widehat{TM}}([v_m]) = \pi(m)$ is a Lie algebroid over \widehat{M} . The fiber of \widehat{TM} over a point $\pi(m) \in \widehat{M}$ is isomorphic to $T_m M$.

The Lie bracket is defined on the space $\Gamma(\tau_{\widehat{TM}})$ which is isomorphic to the Lie subalgebra of G -invariant vector fields, that is,

$$\Gamma(\tau_{\widehat{TM}}) = \{X \in \mathfrak{X}(M) \mid X \text{ is } G\text{-invariant}\}.$$

Thus, the Lie bracket on \widehat{TM} is the bracket of G -invariant vector fields. The anchor map $\rho : \widehat{TM} \rightarrow \widehat{M}$ is given by $\rho([v_m]) = T_m \pi(v_m)$. Moreover, ρ is a Lie algebra homomorphism satisfying the compatibility condition since the G -invariant vector fields are π -projectable. This Lie algebroid is called *Lie-Atiyah algebroid* associated with the principal bundle $\pi : M \rightarrow \widehat{M}$.

Let $\mathcal{A} : TM \rightarrow \mathfrak{g}$ be a principal connection in the principal bundle $\pi : M \rightarrow \widehat{M}$ and $B : TM \oplus TM \rightarrow \mathfrak{g}$ be the curvature of \mathcal{A} . The connection determines an isomorphism $\alpha_{\mathcal{A}}$ between the vector bundles $\widehat{TM} \rightarrow \widehat{M}$ and $T\widehat{M} \oplus \widetilde{\mathfrak{g}} \rightarrow \widehat{M}$, where $\widetilde{\mathfrak{g}} = (M \times \mathfrak{g})/G$ is the adjoint bundle associated with the principal bundle $\pi : M \rightarrow \widehat{M}$ (see [17] for example).

We choose a local trivialization of the principal bundle $\pi : M \rightarrow \widehat{M}$ to be $U \times G$, where U is an open subset of \widehat{M} . Suppose that e is the identity of G , (x^i) are local coordinates on U and $\{\xi_A\}$ is a basis of \mathfrak{g} .

Denote by $\{\xi_A^{\leftarrow}\}$ the corresponding left-invariant vector field on G , that is,

$$\xi_A^{\leftarrow}(g) = (T_e L_g)(\xi_A)$$

for $g \in G$ where $L_g : G \rightarrow G$ is the left-translation on G by g . If

$$\mathcal{A}\left(\frac{\partial}{\partial x^i}\Big|_{(x,e)}\right) = \mathcal{A}_i^A(x)\xi_A, \quad \mathcal{B}\left(\frac{\partial}{\partial x^i}\Big|_{(x,e)}, \frac{\partial}{\partial x^j}\Big|_{(x,e)}\right) = \mathcal{B}_{ij}^A(x)\xi_A,$$

for $i, j \in \{1, \dots, m\}$ and $x \in U$, then the horizontal lift of the vector field $\frac{\partial}{\partial x^i}$ is the vector field on $\pi^{-1}(U) \simeq U \times G$ given by

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - \mathcal{A}_i^A \xi_A^{\leftarrow}.$$

Therefore, the vector fields on $U \times G$

$$e_i = \frac{\partial}{\partial x^i} - \mathcal{A}_i^A \xi_A^{\leftarrow} \text{ and } e_B = \xi_B^{\leftarrow}$$

are G -invariant under the action of G over M and define a local basis $\{\hat{e}_i, \hat{e}_B\}$ on $\Gamma(\widehat{TM}) = \Gamma(\tau_{T\widehat{M} \oplus \widetilde{\mathfrak{g}}})$. The corresponding local structure functions of $\tau_{\widehat{TM}} : \widehat{TM} \rightarrow \widehat{M}$ are

$$\begin{aligned} \mathcal{C}_{ij}^k &= \mathcal{C}_{iA}^j = -\mathcal{C}_{Ai}^j = \mathcal{C}_{AB}^i = 0, & \mathcal{C}_{ij}^A &= -\mathcal{B}_{ij}^A, & \mathcal{C}_{iA}^C &= -\mathcal{C}_{Ai}^C = \mathcal{C}_{AB}^C \mathcal{A}_i^B, \\ \mathcal{C}_{AB}^C &= \mathcal{C}_{AB}^C, & \rho_i^j &= \delta_{ij}, & \rho_i^A &= \rho_A^B = 0, \end{aligned}$$

being $\{c_{AB}^C\}$ the constant structures of \mathfrak{g} with respect to the basis $\{\xi_A\}$ (see [49] for more details). That is,

$$\begin{aligned} \llbracket \hat{e}_i, \hat{x}_j \rrbracket_{\widehat{TM}} &= -\mathcal{B}_{ij}^C \hat{e}_C, \quad \llbracket \hat{e}_i, \hat{e}_A \rrbracket_{\widehat{TM}} = c_{AB}^C \mathcal{A}_i^B \hat{e}_C, \quad \llbracket \hat{e}_A, \hat{e}_B \rrbracket_{\widehat{TM}} = c_{AB}^C \hat{e}_C, \\ \rho_{\widehat{TM}}(\hat{e}_i) &= \frac{\partial}{\partial x^i}, \quad \rho_{\widehat{TM}}(\hat{e}_A) = 0. \end{aligned}$$

The basis $\{\hat{e}_i, \hat{e}_B\}$ induce local coordinates (x^i, y^i, \bar{y}^B) on $\widehat{TM} = TM/G$.

Next, we introduce the notion of Lie subalgebroid associated with a Lie algebroid.

Definition 2.3. Let $(E, \llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ be a Lie algebroid over M and N is a submanifold of M . A Lie subalgebroid of E over N is a vector subbundle B of E over N

$$\begin{array}{ccc} B & \xrightarrow{j} & E \\ \tau_B \downarrow & & \downarrow \tau_E \\ N & \xrightarrow{i} & M \end{array}$$

such that $\rho_B = \rho_E|_B : B \rightarrow TN$ is well define and; given $X, Y \in \Gamma(B)$ and $\tilde{X}, \tilde{Y} \in \Gamma(E)$ arbitrary extensions of X, Y respectively, we have that $(\llbracket \tilde{X}, \tilde{Y} \rrbracket_E)|_N \in \Gamma(B)$.

2.1.2. Examples of Lie subalgebroids.

Example 5. Let E be a Lie algebroid over M . Given a submanifold N of M , if $B = E|_N \cap (\rho|_N)^{-1}(TN)$ exists as a vector bundle, it will be a Lie subalgebroid of E over N , and will be called *Lie algebroid restriction of E to N* (see [51]).

Example 6. Let N be a submanifold of M . Then, TN is a Lie subalgebroid of TM .

Now, let \mathcal{F} be a completely integrable distribution on a manifold M . \mathcal{F} equipped with the bracket of vector fields is a Lie algebroid over M since $\tau_E|_{\mathcal{F}} : \mathcal{F} \rightarrow M$ is a vector bundle and if \mathcal{F} is a foliation, $(\Gamma(\mathcal{F}), [\cdot, \cdot])$ is a Lie algebra. The anchor map is the inclusion $i_{\mathcal{F}} : \mathcal{F} \rightarrow TM$ ($i_{\mathcal{F}}$ is a Lie algebroid monomorphism).

Moreover, if N is a submanifold of M and \mathcal{F}_N is a foliation on N , then \mathcal{F}_N is a Lie subalgebroid of the Lie algebroid $\tau_{TM} : TM \rightarrow M$.

Example 7. Let \mathfrak{g} be a Lie algebra and \mathfrak{h} be a Lie subalgebra. If we consider the Lie algebroid induced by \mathfrak{g} and \mathfrak{h} over a point, then \mathfrak{h} is a Lie subalgebroid of \mathfrak{g} .

Example 8. Let $M \times \mathfrak{g} \rightarrow M$ be an action Lie algebroid and let N be a submanifold of M . Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} such that the infinitesimal generators of the elements of \mathfrak{h} are tangent to N ; that is, the application

$$\begin{aligned} \mathfrak{h} &\rightarrow \mathfrak{X}(N) \\ \xi &\mapsto \xi_N \end{aligned}$$

is well defined. Thus, the action Lie algebroid $N \times \mathfrak{h} \rightarrow N$ is a Lie subalgebroid of $M \times \mathfrak{g} \rightarrow M$.

Example 9. Suppose that the Lie group G acts free and properly on M . Let $\pi : M \rightarrow M/G = \widehat{M}$ be the associated G -principal bundle. Let N be a G -invariant submanifold of M and \mathcal{F}_N be a G -invariant foliation over N . We may consider the vector bundle $\widehat{\mathcal{F}}_N = \mathcal{F}_N/G \rightarrow N/G = \widehat{N}$ and endow it with a Lie algebroid structure. The sections of $\widehat{\mathcal{F}}_N$ are

$$\Gamma(\widehat{\mathcal{F}}_N) = \{X \in \mathfrak{X}(N) \mid X \text{ is } G\text{-invariant and } X(q) \in \mathcal{F}_N(q), \forall q \in N\}.$$

The standard bracket of vector fields on N induces a Lie algebra structure on $\Gamma(\widehat{\mathcal{F}}_N)$. The anchor map is the canonical inclusion of $\widehat{\mathcal{F}}_N$ on $T\widehat{N}$ and $\widehat{\mathcal{F}}_N$ is a Lie subalgebroid of $\widehat{TM} \rightarrow \widehat{M}$.

2.2. E -tangent bundle to a Lie algebroid E . We consider the prolongation over the canonical projection of the Lie algebroid E over M , that is,

$$\mathcal{T}^{\tau_E} E = \bigcup_{e \in E} (E_\rho \times_{T\tau_E} T_e E) = \bigcup_{e \in E} \{(e', v_e) \in E \times T_e E \mid \rho(e') = (T_e \tau_E)(v_e)\},$$

where $T\tau_E : TE \rightarrow TM$ is the tangent map to τ_E .

In fact, $\mathcal{T}^{\tau_E} E$ is a Lie algebroid of rank $2n$ over E where $\tau_E^{(1)} : \mathcal{T}^{\tau_E} E \rightarrow E$ is the vector bundle projection, $\tau_E^{(1)}(b, v_e) = \tau_{TE}(v_e) = e$, and the anchor map is $\rho_1 : \mathcal{T}^{\tau_E} E \rightarrow TE$ is given by the projection over the second factor. The bracket of sections of this new Lie algebroid will be denoted by $[\![\cdot, \cdot]\!]_{\tau_E^{(1)}}$ (See [57] for more details).

If we denote by (e, e', v_e) an element $(e', v_e) \in \mathcal{T}^{\tau_E} E$ where $e \in E$ and where v is tangent; we rewrite the definition for the prolongation of the Lie algebroid as the subset of $E \times E \times TE$ by

$$\mathcal{T}^{\tau_E} E = \{(e, e', v_e) \in E \times E \times TE \mid \rho(e') = (T\tau_E)(v_e), v_e \in T_e E \text{ and } \tau_E(e) = \tau_E(e')\}.$$

Thus, if $(e, e', v_e) \in \mathcal{T}^{\tau_E} E$; then $\rho_1(e, e', v_e) = (e, v_e) \in T_e E$, and $\tau_E^{(1)}(e, e', v_e) = e \in E$.

Next, we introduce two canonical operations that we have on a Lie algebroid E . The first one is obtained using the Lie algebroid structure of E and the second one is a consequence of E being a vector bundle. On one hand, if $f \in C^\infty(M)$ we will denote by f^c the *complete lift* to E of f defined by $f^c(e) = \rho(e)(f)$ for all $e \in E$. Let X be a section of E then there exists a unique vector field X^c on E , the *complete lift* of X , satisfying the two following conditions:

- (1) X^c is τ_E -projectable on $\rho(X)$ and
- (2) $X^c(\hat{\alpha}) = \widehat{\mathcal{L}_X^E \alpha}$,

for every $\alpha \in \Gamma(\tau_E^*)$ (see [33]). Here, if $\beta \in \Gamma(\tau_E^*)$ then $\hat{\beta}$ is the linear function on E defined by

$$\hat{\beta}(e) = \langle \beta(\tau_E(e)), e \rangle, \quad \text{for all } e \in E.$$

We may introduce the *complete lift* X^c of a section $X \in \Gamma(\tau_E)$ as the sections of $\tau_E^{(1)} : \mathcal{T}^{\tau_E} E \rightarrow E$ given by

$$X^c(e) = (X(\tau_E(e)), X^c(e)) \quad (9)$$

for all $e \in E$ (see [57]).

Given a section $X \in \Gamma(\tau_E)$ we define the *vertical lift* as the vector field $X^v \in \mathfrak{X}(E)$ given by

$$X^v(e) = X(\tau_E(e))_e^v, \quad \text{for } e \in E,$$

where $_e^v : E_q \rightarrow T_e E_q$ for $q = \tau_E(e)$ is the canonical isomorphism between the vector spaces E_q and $T_e E_q$.

Finally we may introduce the *vertical lift* X^\vee of a section $X \in \Gamma(\tau_E)$ as a section of $\tau_E^{(1)}$ given by

$$X^\vee(e) = (0, X^v(e)) \text{ for } e \in E.$$

With these definitions we have the properties (see [33] and [57])

$$[X^c, Y^c] = [\![X, Y]\!]^c, \quad [X^c, Y^v] = [\![X, Y]\!]^v, \quad [X^v, Y^v] = 0 \quad (10)$$

for all $X, Y \in \Gamma(\tau_E)$.

If (x^i) are local coordinates on an open subset U of M and $\{e_A\}$ is a basis of sections of τ_E then we have induced coordinates (x^i, y^A) on E . From the basis $\{e_A\}$ we may define a local basis $\{e_A^{(1)}, e_A^{(2)}\}$ of sections of $\tau_E^{(1)}$ given by

$$e_A^{(1)}(e) = \left(e, e_A(\tau_A(e)), \rho_A^i \frac{\partial}{\partial x^i} \Big|_e \right), \quad e_A^{(2)}(e) = \left(e, 0, \frac{\partial}{\partial y^A} \Big|_e \right),$$

for $e \in (\tau_E)^{-1}(U)$ with U an open subset of M (see [49] for more details).

From this basis we have that the structure of Lie algebroid is determined by

$$\begin{aligned}\rho_1(e_A^{(1)}(e)) &= \left(e, \rho_A^i \frac{\partial}{\partial x^i} \Big|_e \right), & \rho_1(e_A^{(2)}(e)) &= \left(e, \frac{\partial}{\partial y^A} \Big|_e \right) \\ \llbracket e_A^{(1)}, e_B^{(1)} \rrbracket_{\tau_E^{(1)}} &= C_{AB}^C e_C^{(1)}, \\ \llbracket e_A^{(1)}, e_B^{(2)} \rrbracket_{\tau_E^{(1)}} &= \llbracket e_A^{(2)}, e_B^{(2)} \rrbracket_{\tau_E^{(1)}} = 0,\end{aligned}$$

for all A, B and C ; where C_{AB}^C are the structure functions of E determined by the Lie bracket $\llbracket \cdot, \cdot \rrbracket$ with respect to the basis $\{e_A\}$.

Using $\{e_A^{(1)}, e_A^{(2)}\}$ one may introduce local coordinates $(x^i, y^A; z^A, v^A)$ on E . If V is a section of $\tau_E^{(1)}$, locally it is determined by

$$V(x, y) = (x^i, y^A, z^A(x, y), v^A(x, y));$$

therefore the expression of V in terms of the basis $\{e_A^{(1)}, e_A^{(2)}\}$ is $V = z^A e_A^{(1)} + v^A e_A^{(2)}$ and the vector field $\rho_1(V) \in \mathfrak{X}(E)$ has the expression

$$\rho_1(V) = \rho_A^i z^A(x, y) \frac{\partial}{\partial x^i} \Big|_{(x, y)} + v^A(x, y) \frac{\partial}{\partial y^A} \Big|_{(x, y)}.$$

Moreover, if $\{e_{(1)}^A, e_{(2)}^A\}$ denotes the dual basis of $\{e_A^{(1)}, e_A^{(2)}\}$,

$$\begin{aligned}d^{\mathcal{T}^{\tau^E E}} F(x^i, y^A) &= \rho_A^i \frac{\partial F}{\partial x^i} e_{(1)}^A + \frac{\partial F}{\partial y^A} e_{(2)}^A, \\ d^{\mathcal{T}^{\tau^E E}} e_{(1)}^C &= -\frac{1}{2} C_{AB}^C e_{(1)}^A \wedge e_{(1)}^B, \quad d^{\mathcal{T}^{\tau^E E}} e_{(2)}^C = 0.\end{aligned}$$

Example 10. In the case of $E = TM$ one may identify $\mathcal{T}^{\tau^E E}$ with TTM with the standard Lie algebroid structure over TM .

Example 11. Let \mathfrak{g} be a real Lie algebra of finite dimension. \mathfrak{g} is a Lie algebroid over a single point $M = \{q\}$. The anchor map of \mathfrak{g} is zero constant function, and from the anchor map we deduce that

$$\mathcal{T}^{\tau^{\mathfrak{g}}} \mathfrak{g} = \{(\xi_1, \xi_2, v_{\xi_1}) \in \mathfrak{g} \times T\mathfrak{g}\} \simeq \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \simeq 3\mathfrak{g}.$$

The vector bundle projection $\tau_{\mathfrak{g}}^{(1)} : 3\mathfrak{g} \rightarrow \mathfrak{g}$ is given by $\tau_{\mathfrak{g}}^{(1)}(\xi_1, \xi_2, \xi_3) = \xi_1$ with anchor map $\rho_1(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_3) \simeq v_{\xi_1} \in T_{\xi_1} \mathfrak{g}$.

Let $\{e_A\}$ be a basis of the Lie algebra \mathfrak{g} , this basis induces local coordinates y^A on \mathfrak{g} , that is, $\xi = y^A e_A$. Also, this basis induces a basis of sections of $\tau_{\mathfrak{g}}^{(1)}$ as

$$e_A^{(1)}(\xi) = (\xi, e_A, 0), \quad e_A^{(2)}(\xi) = \left(\xi, 0, \frac{\partial}{\partial y^A} \right).$$

Moreover

$$\rho_1(e_A^{(1)})(\xi) = (\xi, 0), \quad \rho_1(e_A^{(2)})(\xi) = \left(\xi, \frac{\partial}{\partial y^A} \right).$$

The basis $\{e_A^{(1)}, e_A^{(2)}\}$ induces adapted coordinates (y^A, z^A, v^A) in $3\mathfrak{g}$ and therefore a section Y on $\Gamma(\tau_{\mathfrak{g}}^{(1)})$ is written as $Y(\xi) = z^A(\xi) e_A^{(1)} + v^A(\xi) e_A^{(2)}$. Thus, the vector field $\rho_1(Y) \in \mathfrak{g}$ has the expression $\rho_1(Y) = v^A(\xi) \frac{\partial}{\partial y^A} \Big|_{\xi}$. Finally, the Lie algebroid structure on $\tau_{\mathfrak{g}}^{(1)}$ is determined by the Lie bracket $\llbracket (\xi, \tilde{\xi}), (\eta, \tilde{\eta}) \rrbracket = ([\xi, \eta], 0)$.

Example 12. We consider a Lie algebra \mathfrak{g} acting on a manifold M , that is, we have a Lie algebra homomorphism $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ mapping every element ξ of \mathfrak{g} to a vector field ξ_M on M . Then we can consider the action Lie algebroid $E = M \times \mathfrak{g}$. Identifying $TE = TM \times T\mathfrak{g} = TM \times 2\mathfrak{g}$, an element of the prolongation Lie algebroid to E over the bundle projection is of the form $(a, b, v_a) = ((x, \xi), (x, \eta), (v_x, \xi, \chi))$ where $x \in M$, $v_x \in T_x M$ and

$(\xi, \eta, \chi) \in 3\mathfrak{g}$. The condition $T\tau_{\mathfrak{g}}(v) = \rho(b)$ implies that $v_x = -\eta_M(x)$. Therefore we can identify the prolongation Lie algebroid with $M \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ with projection onto the first two factors (x, ξ) and anchor map $\rho_1(x, \xi, \eta, \chi) = (-\eta_M(x), \xi, \chi)$. Given a base $\{e_A\}$ of \mathfrak{g} the basis $\{e_A^{(1)}, e_A^{(2)}\}$ of sections of $\mathcal{T}^{\tau_{\mathfrak{g}} \times \mathfrak{g}}(M \times \mathfrak{g})$ is given by

$$e_A^{(1)}(x, \xi) = (x, \xi, e_A, 0), \quad e_A^{(2)}(x, \xi) = (x, \xi, 0, e_A).$$

Moreover,

$$\rho_1(e_A^{(1)}(x, \xi)) = (x, -(e_A)_M(x), \xi, 0), \quad \rho_2(e_A^{(2)}(x, \xi)) = (x, 0, \xi, e_A).$$

Finally, the Lie bracket of two constant sections is given by $\llbracket (\xi, \tilde{\xi}), (\eta, \tilde{\eta}) \rrbracket = ([\xi, \eta], 0)$.

Example 13. Let us describe the E -tangent bundle to E in the case of E being an Atiyah algebroid induced by a trivial principal G -bundle $\pi : G \times M \rightarrow M$. In such case, by left trivialization we get the Atiyah algebroid, the vector bundle $\tau_{\mathfrak{g} \times TM} : \mathfrak{g} \times TM \rightarrow TM$. If $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{g}$ then we may consider the section $X^\xi : M \rightarrow \mathfrak{g} \times TM$ of the Atiyah algebroid by

$$X^\xi(q) = (\xi, X(q)) \text{ for } q \in M.$$

Moreover, in this sense

$$\llbracket X^\xi, Y^\xi \rrbracket_{\mathfrak{g} \times TM} = ([X, Y]_{TM}, [\xi, \eta]_{\mathfrak{g}}), \quad \rho(X^\xi) = X.$$

On the other hand, if $(\xi, v_q) \in \mathfrak{g} \times T_q M$, then the fiber of $\mathcal{T}^{\tau_{\mathfrak{g}} \times TM}(\mathfrak{g} \times TM)$ over (ξ, v_q) is

$$\mathcal{T}_{(\xi, v_q)}^{\tau_{\mathfrak{g}} \times TM}(\mathfrak{g} \times TM) = \left\{ ((\eta, u_q), (\tilde{\eta}, X_{v_q})) \in \mathfrak{g} \times T_q M \times \mathfrak{g} \times T_{v_q}(TM) \right. \\ \left. \text{such that } u_q = T_{v_q} \tau_{\mathfrak{g} \times TM}(X_{v_q}) \right\}.$$

This implies that $\mathcal{T}_{(\xi, v_q)}^{\tau_{\mathfrak{g}} \times TM}(\mathfrak{g} \times TM)$ may be identified with the space $2\mathfrak{g} \times T_{v_q}(TM)$. Thus, the Lie algebroid $\mathcal{T}^{\tau_{\mathfrak{g}} \times TM}(\mathfrak{g} \times TM)$ may be identified with the vector bundle $\mathfrak{g} \times 2\mathfrak{g} \times TTM \rightarrow \mathfrak{g} \times TM$ whose vector bundle projection is

$$(\xi, ((\eta, \tilde{\eta}), X_{v_q})) \mapsto (\xi, v_q)$$

for $(\xi, ((\eta, \tilde{\eta}), X_{v_q})) \in \mathfrak{g} \times 2\mathfrak{g} \times TTM$. Therefore, if $(\eta, \tilde{\eta}) \in 2\mathfrak{g}$ and $X \in \mathfrak{X}(TM)$ then one may consider the section $((\eta, \tilde{\eta}), X)$ given by

$$((\eta, \tilde{\eta}), X)(\xi, v_q) = (\xi, ((\eta, \tilde{\eta}), X(v_q))) \text{ for } (\xi, v_q) \in \mathfrak{g} \times T_q M.$$

Moreover,

$$\llbracket ((\eta, \tilde{\eta}), X), ((\xi, \tilde{\xi}), Y) \rrbracket_{\tau_{\mathfrak{g} \times TM}^{(1)}} = (([\eta, \xi]_{\mathfrak{g}}, 0), [X, Y]_{TM}),$$

and the anchor map $\rho_1 : \mathfrak{g} \times 2\mathfrak{g} \times TTM \rightarrow \mathfrak{g} \times \mathfrak{g} \times TTM$ is defined as

$$\rho_1(\xi, ((\eta, \tilde{\eta}), X)) = ((\xi, \tilde{\eta}), X).$$

2.3. E -tangent bundle of the dual bundle of a Lie algebroid. Let $(E, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid of rank n over a manifold of dimension m . Consider the projection of the dual E^* of E over M , $\tau_{E^*} : E^* \rightarrow M$, and define the prolongation $\mathcal{T}^{\tau_{E^*}} E$ of E over τ_{E^*} ; that is,

$$\mathcal{T}^{\tau_{E^*}} E = \bigcup_{\mu \in E^*} \{(e, v_\mu) \in E \times T_\mu E^* \mid \rho(e) = T\tau_{E^*}(v_\mu)\}.$$

$\mathcal{T}^{\tau_{E^*}} E$ is a Lie algebroid over E^* of rank $2n$ with vector bundle projection $\tau_{E^*}^{(1)} : \mathcal{T}^{\tau_{E^*}} E \rightarrow E^*$ given by $\tau_{E^*}^{(1)}(e, v_\mu) = \mu$, for $(e, v_\mu) \in \mathcal{T}^{\tau_{E^*}} E$.

As before, if we now denote by (μ, e, v_μ) an element $(e, v_\mu) \in \mathcal{T}^{\tau_{E^*}} E$ where $\mu \in E^*$, we rewrite the definition of the prolongation Lie algebroid as the subset of $E^* \times E \times TE^*$ by $\mathcal{T}^{\tau_{E^*}} E = \{(\mu, e, v_\mu) \in E^* \times E \times TE^* \mid \rho(e) = (T\tau_{E^*})(v_\mu), v_\mu \in T_\mu E^* \text{ and } \tau_{E^*}(\mu) = \tau_E(e)\}.$

If (x^i) are local coordinates on an open subset U of M , $\{e_A\}$ is a basis of sections of the vector bundle $(\tau_E)^{-1}(U) \rightarrow U$ and $\{e^A\}$ is its dual basis, then $\{\tilde{e}_A^{(1)}, \tilde{e}_A^{(2)}\}$ is a basis of sections of the vector bundle $\tau_{E^*}^{(1)}$, where

$$\tilde{e}_A^{(1)}(\mu) = \left(\mu, e_A(\tau_{E^*}(\mu)), \rho_A^i \frac{\partial}{\partial x^i} \Big|_\mu \right), \quad (\tilde{e}^A)^{(2)}(\mu) = \left(\mu, 0, \frac{\partial}{\partial p_A} \Big|_\mu \right),$$

for $\mu \in (\tau_{E^*})^{-1}(U)$. Here, (x^i, p_A) are the local coordinates on E^* induced by the local coordinates (x^i) and the basis of sections of E^* , $\{e^A\}$.

Using the local basis $\{\tilde{e}_A^{(1)}, (\tilde{e}^A)^{(2)}\}$, one may introduce, in a natural way, local coordinates $(x^i, p_A; z^A, v_A)$ on $\mathcal{T}^{\tau_{E^*}} E$. If ω^* is a point of $\mathcal{T}^{\tau_{E^*}} E$ over $(x, p) \in E^*$, then

$$\omega^*(x, p) = z^A \tilde{e}_A^{(1)}(x, p) + v_A (\tilde{e}^A)^{(2)}(x, p).$$

Denoting by $\rho_{\tau_{E^*}^{(1)}}$ the anchor map of the Lie algebroid $\mathcal{T}^{\tau_{E^*}} E \rightarrow E^*$ locally given by

$$\rho_{\tau_{E^*}^{(1)}}(x^i, p_A, z^A, v_A) = (x^i, p_A, \rho_A^i z^A, v^A),$$

we have that

$$\rho_{\tau_{E^*}^{(1)}}(\tilde{e}_A^{(1)})(\mu) = \left(\mu, \rho_A^i \frac{\partial}{\partial x^i} \Big|_\mu \right), \quad \rho_{\tau_{E^*}^{(1)}}((\tilde{e}^A)^{(2)})(\mu) = \left(\mu, \frac{\partial}{\partial p_A} \Big|_\mu \right).$$

Therefore, we have that the corresponding vector field $\rho_{\tau_{E^*}^{(1)}}(V)$ for the section determined by $V = (x^i, p_A, z^A(x, p), v_A(x, p))$ is given by

$$\rho_{\tau_{E^*}^{(1)}}(V) = \rho_A^i z^A \frac{\partial}{\partial x^i} \Big|_\mu + v_A \frac{\partial}{\partial p_A} \Big|_{e^*}.$$

Finally, the structure of the Lie algebroid $(\mathcal{T}^{\tau_{E^*}} E, [\cdot, \cdot]_{\tau_{E^*}^{(1)}}, \rho_{\tau_{E^*}^{(1)}})$, is determined by the bracket of sections

$$[\tilde{e}_A^{(1)}, \tilde{e}_B^{(1)}]_{\tau_{E^*}^{(1)}} = \mathcal{C}_{AB}^C \tilde{e}_C^{(1)}, \quad [(\tilde{e}^A)^{(2)}, (\tilde{e}^B)^{(2)}]_{\tau_{E^*}^{(1)}} = [(\tilde{e}^A)^{(2)}, (\tilde{e}^B)^{(2)}]_{\tau_{E^*}^{(1)}} = 0,$$

for all A, B and C . Thus, if we denote by $\{\tilde{e}_{(1)}^A, (\tilde{e}_A)_{(2)}\}$ is the dual basis of $\{\tilde{e}_A^{(1)}, (\tilde{e}^A)^{(2)}\}$, then

$$\begin{aligned} d^{\mathcal{T}^{\tau_{E^*}} E} f(x^i, p_A) &= \rho_A^i \frac{\partial f}{\partial x^i} \tilde{e}_{(1)}^A + \frac{\partial f}{\partial p_A} (\tilde{e}_A)_{(2)}, \\ d^{\mathcal{T}^{\tau_{E^*}} E} \tilde{e}_{(1)}^C &= -\frac{1}{2} \mathcal{C}_{AB}^C \tilde{e}_{(1)}^A \wedge \tilde{e}_{(1)}^B, \quad d^{\mathcal{T}^{\tau_{E^*}} E} (\tilde{e}_C)_{(2)} = 0, \end{aligned}$$

for $f \in C^\infty(E^*)$. We refer to [49] for further details about the Lie algebroid structure of the E-tangent bundle of the dual bundle of a Lie algebroid.

Example 14. In the case of $E = TM$ one may identify $\mathcal{T}^{\tau_{E^*}} E$ with $T(T^*M)$ with the standard Lie algebroid structure.

Example 15. Let \mathfrak{g} be a real Lie algebra of finite dimension. Then \mathfrak{g} is a Lie algebroid over a single point. Using that the anchor map is zero we have that $\mathcal{T}^{\tau_{\mathfrak{g}^*}} \mathfrak{g}$ may be identified with the vector bundle $pr_1 : \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \rightarrow \mathfrak{g}^*$. Under this identification the anchor map is given by

$$\rho_{\tau_{\mathfrak{g}^*}^{(1)}} : \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \rightarrow T\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}^*, \quad (\mu, (\xi, \alpha)) \mapsto (\mu, \alpha)$$

and the Lie bracket of two constant sections $(\xi, \alpha), (\eta, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$ is the constant section $([\xi, \eta], 0)$.

Example 16. Let $A = M \times \mathfrak{g} \rightarrow M$ be an action Lie algebroid over M and $(q, \mu) \in M \times \mathfrak{g}^*$. It follows that the prolongation may be identified with the trivial vector bundle $(M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \rightarrow M \times \mathfrak{g}^*$ since

$$\mathcal{T}^{\tau(M \times \mathfrak{g}^*)} M \times \mathfrak{g} = \{((q, \xi), (X_q, \alpha)) \in M \times \mathfrak{g} \times T_q M \times \mathfrak{g}^* \mid -\xi_M(q) = X_q\} \simeq \mathfrak{g} \times \mathfrak{g}^*.$$

The anchor map $\rho_{\tau_{M \times \mathfrak{g}^*}^{(1)}} : (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \rightarrow TM \times (\mathfrak{g}^* \times \mathfrak{g}^*)$ is given by

$$\rho_{\tau_{M \times \mathfrak{g}^*}^{(1)}}((q, \mu), (\xi, \alpha)) = (-\xi_M(q), \mu, \alpha).$$

Moreover, the Lie bracket of two constant sections $(\xi, \alpha), (\eta, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$ is just the constant section $([\xi, \eta], 0)$.

Example 17. Let us describe the A -tangent bundle to A in the case of A being an Atiyah algebroid induced by a trivial principal G -bundle $\pi : G \times M \rightarrow M$. In such case, by left trivialization we have that the Atiyah algebroid is the vector bundle $\tau_{\mathfrak{g} \times TM} : \mathfrak{g} \times TM \rightarrow TM$. If $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{g}$ then we may consider the section $X^\xi : M \rightarrow \mathfrak{g} \times TM$ of the Atiyah algebroid by

$$X^\xi(q) = (\xi, X(q)) \text{ for } q \in M.$$

Moreover, in this sence

$$[[X^\xi, Y^\xi]]_{\mathfrak{g} \times TM} = ([X, Y]_{TM}, [\xi, \eta]_{\mathfrak{g}}), \quad \rho(X^\xi) = X.$$

If $(\mu, \alpha_q) \in \mathfrak{g}^* \times T_q^* M$ then the fiber of $\mathcal{T}^{\tau(\mathfrak{g} \times TM)^*}(\mathfrak{g} \times TM)$ over (μ, α_q) is

$$\begin{aligned} \mathcal{T}_{(\mu, \alpha_q)}^{\tau(\mathfrak{g} \times TM)^*}(\mathfrak{g} \times TM) = & \left\{ ((\eta, u_q), (\beta, X_{\alpha_q})) \in \mathfrak{g} \times T_q M \times \mathfrak{g}^* \times T_{v_q}(T^* M) \right. \\ & \left. \text{such that } u_q = T_{\alpha_q} \tau_{(\mathfrak{g} \times TM)^*}(X_{\alpha_q}) \right\}. \end{aligned}$$

This implies that $\mathcal{T}_{(\mu, \alpha_q)}^{\tau(\mathfrak{g} \times TM)^*}(\mathfrak{g} \times TM)$ may be identified with the vector space $(\mathfrak{g} \times \mathfrak{g}^*) \times T_{\alpha_q}(T^* M)$. Thus, the Lie algebroid $\mathcal{T}^{\tau(\mathfrak{g} \times TM)^*}(\mathfrak{g} \times TM)$ may be identified with the vector bundle $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times TT^* M \rightarrow \mathfrak{g}^* \times T^* M$ whose vector bundle projection is

$$(\mu, ((\xi, \beta), X_{\alpha_q})) \mapsto (\mu, \alpha_q)$$

for $(\mu, ((\xi, \beta), X_{\alpha_q})) \in \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times TT^* M$. Therefore, if $(\xi, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$ and $X \in \mathfrak{X}(T^* M)$ then one may consider the section $((\xi, \beta), X)$ given by

$$((\xi, \beta), X)(\mu, \alpha_q) = (\mu, ((\xi, \beta), X(\alpha_q))) \text{ for } (\mu, \alpha_q) \in \mathfrak{g}^* \times T_q^* M.$$

Moreover,

$$[[((\xi, \beta), X), ((\tilde{\xi}, \tilde{\beta}), \tilde{X})]]_{\tau_{(\mathfrak{g} \times TM)^*}^{(1)}} = (([\xi, \tilde{\xi}]_{\mathfrak{g}}, 0), [X, \tilde{X}]_{TM}),$$

and the anchor map $\rho_{\tau_{(\mathfrak{g} \times TM)^*}^{(1)}} : \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times TT^* M \rightarrow \mathfrak{g}^* \times \mathfrak{g}^* \times TT^* M$ is defined as

$$\rho_{\tau_{(\mathfrak{g} \times TM)^*}^{(1)}}(\mu, ((\xi, \beta), X)) = ((\mu, \beta), X).$$

2.4. Symplectic Lie algebroids. In this subsection we will recall some results given in [49] about symplectic Lie algebroids.

Definition 2.4. A Lie algebroid $(E, [\cdot, \cdot], \rho)$ over a manifold M is said to be symplectic if it admits a symplectic section Ω , that is, Ω is a section of the vector bundle $\bigwedge^2 E^* \rightarrow M$ such that:

- (1) For all $x \in M$, the 2-form $\Omega_x : E_x \times E_x \rightarrow \mathbb{R}$ in the vector space E_x is nondegenerate and
- (2) Ω is a 2-cocycle, that is, $d^E \Omega = 0$.

2.4.1. *The canonical symplectic structure of $\mathcal{T}^{\tau E^*} E$.* Let $(E, [\cdot, \cdot], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m and $\mathcal{T}^{\tau E^*} E$ be the prolongation of E over the vector bundle projection $\tau_{E^*} : E^* \rightarrow M$. We may introduce a canonical section λ_E of $(\mathcal{T}^{\tau E^*} E)^*$ as follows. If $\mu \in E^*$ and (e, v_μ) is a point on the fibre of $\mathcal{T}^{\tau E^*} E$ over μ then

$$\lambda_E(\mu)(e, v_\mu) = \langle \mu, e \rangle. \quad (11)$$

λ_E is called the *Liouville section* of $\mathcal{T}^{\tau E^*} E$. Now, in an analogous way that the canonical symplectic form is defined from the Liouville 1-form on the cotangent bundle, we introduce the 2-section Ω_E on $\mathcal{T}^{\tau E^*} E$ as

$$\Omega_E = -d^{\mathcal{T}^{\tau E^*} E} \lambda_E. \quad (12)$$

Proposition 1. [49] Ω_E is a non-degenerate 2-section of $\mathcal{T}^{\tau E^*} E$ such that

$$d^{\mathcal{T}^{\tau E^*} E} \Omega_E = 0.$$

Therefore Ω_E is a symplectic 2-section on $\mathcal{T}^{\tau E^*} E$ called *canonical symplectic section* on $\mathcal{T}^{\tau E^*} E$.

Example 18. If E is the standard Lie algebroid TM then $\lambda_E = \lambda$ and $\Omega_E = \omega_M$ are the usual Liouville 1-form and canonical symplectic 2-form on T^*M , respectively.

Example 19. Let \mathfrak{g} be a finite dimensional Lie algebra. Then \mathfrak{g} is a Lie algebroid over a single point $M = \{q\}$. If $\xi \in \mathfrak{g}$ and $\mu, \alpha \in \mathfrak{g}^*$ then

$$\lambda_{\mathfrak{g}}(\mu)(\xi, \alpha) = \mu(\xi)$$

is the Liouville 1-section on $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*)$. Thus, the symplectic section $\Omega_{\mathfrak{g}}$ is

$$\Omega_{\mathfrak{g}}(\mu)((\xi, \alpha), (\eta, \beta)) = \langle \mu, [\xi, \eta] \rangle - \langle \alpha, \eta \rangle - \langle \beta, \xi \rangle$$

for $\mu \in \mathfrak{g}^*, (\xi, \alpha), (\eta, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$.

2.5. Admissible elements on a Lie algebroid. Let E be a Lie algebroid over M with fiber bundle projection $\tau_E : E \rightarrow M$ and anchor map $\rho : E \rightarrow TM$.

Definition 2.5. A tangent vector v at the point $e \in E$ is called *admissible* if $\rho(e) = T_e \tau_E(v)$; and a curve on E is admissible if its tangent vectors are admissible. The set of admissible elements on E will be denote $E^{(2)}$.

Notice that v is admissible if and only if (e, e, v) is in $\mathcal{T}^{\tau E} E$. We will consider $E^{(2)}$ as the subset of the prolongation of E over τ_E , that is, $E^{(2)} \subset E_\rho \times_{T\tau_E} TE$ is given by

$$E^{(2)} = \{(e, v_e) \in E \times TE \mid \rho(e) = T\tau_E(v_e)\}.$$

Other authors call this set $\text{Adm}(E)$ (see [14] and [57]).

A curve $\sigma : I \subset \mathbb{R} \rightarrow E$ is said to be an admissible curve on E if it satisfies $\rho(\sigma(t)) = \dot{\gamma}(t)$ where $\gamma = \tau_E(\sigma(t))$ is a curve on M . Locally, admissible curves on E are characterized by the so-called admissibility condition. A curve $\gamma(t) = (x^i(t), y^A(t))$ on E is admissible if it satisfies the admissibility condition $\dot{x}^i(t) = \rho_A^i(x^i(t))y^A(t)$. Therefore, locally, $E^{(2)}$ is determined by $(\gamma(0), \dot{\gamma}(0))$ where γ is an admissible curve on E . Admissible curves on E are also called *E-path* [58].

We consider $E^{(2)}$ as the substitute of the second order tangent bundle in classical mechanics. If (x^i) are local coordinates on M and $\{e_A\}$ is a basis of sections of E then we denote (x^i, y^A) the corresponding local coordinates on E and (x^i, y^A, z^A, v^A) local coordinates on $\mathcal{T}^{\tau E} E$ induced by the basis of sections $\{e_A^{(1)}, e_A^{(2)}\}$ of $\mathcal{T}^{\tau E} E$ (see subsection 2.2). Therefore, the set $E^{(2)}$ is locally characterized by the condition $\{(x^i, y^A, z^A, v^A) \in \mathcal{T}^{\tau E} E \mid y^A = z^A\}$, that is $(x^i, y^A, v^A) := (x^i, y^A, \dot{y}^A)$ are local coordinates on $E^{(2)}$.

We denote the canonical inclusion of $E^{(2)}$ on the prolongation of E over τ_E as

$$\begin{aligned} i_{E^{(2)}} : E^{(2)} &\hookrightarrow \mathcal{T}^{\tau E} E, \\ (x^i, y^A, \dot{y}^A) &\longmapsto (x^i, y^A, y^A, \dot{y}^A). \end{aligned}$$

Example 20. Let M be a differentiable manifold of dimension m , if (x^i) are local coordinates on M , then $\{\frac{\partial}{\partial x^i}\}$ is a local basis of $\mathfrak{X}(M)$ and then we have fiber local coordinates (x^i, \dot{x}^i) on TM . The corresponding local structure functions of the Lie algebroid $\tau_{TM} : TM \rightarrow M$ are

$$C_{ij}^k = 0 \text{ and } \rho_i^j = \delta_i^j, \text{ for } i, j, k \in \{1, \dots, m\}.$$

In this case, we have seen that the prolongation Lie algebroid over τ_{TM} is just the tangent bundle $T(TM)$ where the Lie algebroid structure of the vector bundle $T(TM) \rightarrow TM$ is as we have described above as the tangent bundle of a manifold.

The set of admissible elements is given by

$$E^{(2)} = \{(x^i, v^i, \dot{x}^i, w^i) \in T(TM) \mid \dot{x}^i = v^i\}$$

and observe that this subset is just the second-order tangent bundle of a manifold M , that is, $E^{(2)} = T^{(2)}M$. Admissible curves on $E^{(2)} = T^{(2)}M$ are given by

$$\sigma(t) = (x^i(t), \dot{x}^i(t), \ddot{x}^i(t)).$$

Example 21. Consider a Lie algebra \mathfrak{g} as a Lie algebroid over a point $\{e\}$. Given a basis of section $\{e_A\}$ and element $\xi \in \mathfrak{g}$ can be written as $\xi = e_A \xi^A$ and given that the anchor map is given by $\rho(\xi) \equiv 0$, every curve $\xi(t) \in \mathfrak{g}$ is an admissible curve. The set of admissible elements is described by the cartesian product of two copies of the Lie algebra, $2\mathfrak{g}$. Local coordinates on $2\mathfrak{g}$ are determined by the basis of sections of \mathfrak{g} , $\{e_A\}$ and $\{e_A^{(1)}, e_A^{(2)}\}$, the basis of the prolongation Lie algebroid introduced in Example 11. They are denoted by (ξ^1, ξ^2) and also $(\xi^1, \xi^2) := (\xi(0), \dot{\xi}(0)) \in 2\mathfrak{g}$ where $\xi(t)$ is admissible.

Example 22. Let G be a Lie group and we assume that G acts free and properly on M . We denote by $\pi : M \rightarrow \widehat{M} = M/G$ the associated principal bundle. The tangent lift of the action gives a free and proper action of G on TM and $\widehat{TM} = TM/G$ is a quotient manifold. Then we consider the Atiyah algebroid \widehat{TM} over \widehat{M} .

According to example 4, the basis $\{\hat{e}_i, \hat{e}_B\}$ induce local coordinates (x^i, y^i, \bar{y}^B) . From this basis one can induce a basis of the prolongation Lie algebroid, namely $\{\hat{e}_i^{(1)}, \hat{e}_B^{(1)}\}$. This basis induce adapted coordinates $(x^i, y^i, \bar{y}^B, \dot{y}^i, \dot{\bar{y}}^B)$ on $\widehat{T^{(2)}M} = (T^{(2)}M)/G$.

3. SECOND-ORDER VARIATIONAL PROBLEMS ON LIE ALGEBROIDS

The geometric description of mechanics in terms of Lie algebroids gives a general framework to obtain all the relevant equations in mechanics (Euler-Lagrange, Euler-Poincaré, Lagrange-Poincaré,...). In this section we use the notion of Lie algebroid and prolongation of a Lie algebroid described in §2 to derive the Euler-Lagrange equations and Hamilton equations on Lie algebroids. Next, after introduce the constraint algorithm for presymplectic Lie algebroids and study vakonomic mechanics on Lie algebroids, we study the geometric formalism for second-order constrained variational problems using and adaptation of the classical Skinner-Rusk formalism for the second-order constrained systems on Lie algebroids.

3.1. Mechanics on Lie algebroids. In [57] (see also [49]) a geometric formalism for Lagrangian mechanics on Lie algebroids was introduced. It was developed in the prolongation $\mathcal{T}^{\tau_E}E$ of a Lie algebroid E (see §2) over the vector bundle projection $\tau_E : E \rightarrow M$. The prolongation of the Lie algebroid is playing the same role as TTQ in the standard mechanics. We first introduce the canonical geometrical structures defined on $\mathcal{T}^{\tau_E}E$ to derive the Euler-Lagrange equations on Lie algebroids.

Two canonical objects on $\mathcal{T}^{\tau_E}E$ are the Euler section Δ and the vertical endomorphism S . Considering the local basis of sections of $\mathcal{T}^{\tau_E}E$, $\{e_A^{(1)}, e_A^{(2)}\}$, Δ is the section of $\mathcal{T}^{\tau_E}E \rightarrow E$ locally defined by

$$\Delta = y^A e_A^{(2)} \quad (13)$$

and S is the section of the vector bundle $(\mathcal{T}^{\tau E} E) \otimes (\mathcal{T}^{\tau E} E)^* \rightarrow E$ locally characterized by the following conditions:

$$S e_A^{(1)} = e_A^{(2)}, \quad S e_A^{(2)} = 0, \quad \text{for all } A. \quad (14)$$

Finally, a section ξ of $\mathcal{T}^{\tau E} E \rightarrow E$ is said to be a *second order differential equation* (SODE) on E if $S(\xi) = \Delta$ or, alternatively, $pr_1(\xi(e)) = e$, for all $e \in E$ (for more details, see [49]).

Given a Lagrangian function $L \in C^\infty(E)$ we introduce the *Cartan 1-section* $\Theta_L \in \Gamma((\mathcal{T}^{\tau E} E)^*)$, the *Cartan 2-section* $\omega_L \in \Gamma(\wedge^2(\mathcal{T}^{\tau E} E)^*)$ and the *Lagrangian energy* $E_L \in C^\infty(E)$ as

$$\Theta_L = S^*(d\mathcal{T}^{\tau E} E L), \quad \omega_L = -d\mathcal{T}^{\tau E} E \Theta_L \quad E_L = \mathcal{L}_\Delta^{\mathcal{T}^{\tau E} E} L - L.$$

If (x^i, y^A) are local fibred coordinates on E , (ρ_A^i, C_{AB}^C) are the corresponding local structure functions on E and $\{e_A^{(1)}, e_A^{(2)}\}$ the corresponding local basis of sections of $\mathcal{T}^{\tau E} E$ then

$$\omega_L = \frac{\partial^2 L}{\partial y^A \partial y^B} e_{(1)}^A \wedge e_{(2)}^B + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^A} \rho_B^i - \frac{\partial^2 L}{\partial x^i \partial y^B} \rho_A^i + \frac{\partial L}{\partial y^A} C_{AB}^C \right) e_{(1)}^A \wedge e_{(1)}^B, \quad (15)$$

$$E_L = \frac{\partial L}{\partial y^A} y^A - L. \quad (16)$$

A curve $t \mapsto c(t)$ on E is a solution of the *Euler-Lagrange equations* for L if

- c is *admissible* (that is, $\rho(c(t)) = \dot{m}(t)$, where $m = \tau_E \circ c$) and
- $i_{(c(t), \dot{c}(t))} \omega_L(c(t)) - d\mathcal{T}^{\tau E} E E_L(c(t)) = 0$, for all t .

If $c(t) = (x^i(t), y^A(t))$ then c is a solution of the Euler-Lagrange equations for L if and only if

$$\dot{x}^i = \rho_A^i y^A, \quad \frac{d}{dt} \frac{\partial L}{\partial y^A} + \frac{\partial L}{\partial y^C} C_{AB}^C y^B - \rho_A^i \frac{\partial L}{\partial x^i} = 0. \quad (17)$$

Observe that, if E is the standard Lie algebroid TQ then the above equations are the classical Euler-Lagrange equations for $L : TQ \rightarrow \mathbb{R}$.

On the other hand, the Lagrangian function L is said to be *regular* if ω_L is a symplectic section. In such a case, there exists a unique solution ξ_L verifying

$$i_{\xi_L} \omega_L - d\mathcal{T}^{\tau E} E E_L = 0. \quad (18)$$

In addition, one can check that $i_{S\xi_L} \omega_L = i_\Delta \omega_L$ which implies that ξ_L is a SODE section. Thus, the integral curves of ξ_L (that is, the integral curves of the vector field $\rho_1(\xi_L)$) are solutions of the Euler-Lagrange equations for L . ξ_L is called the *Euler-Lagrange section* associated with L .

From (15), we deduce that the Lagrangian L is regular if and only if the matrix

$$(W_{AB}) = \left(\frac{\partial^2 L}{\partial y^A \partial y^B} \right)$$

is regular. Moreover, the local expression of ξ_L is

$$\xi_L = y^A e_A^{(1)} + f^A e_A^{(2)},$$

where the functions f^A satisfy the linear equations

$$\frac{\partial^2 L}{\partial y^B \partial y^A} f^B + \frac{\partial^2 L}{\partial x^i \partial y^A} \rho_B^i y^B + \frac{\partial L}{\partial y^C} C_{AB}^C y^B - \rho_A^i \frac{\partial L}{\partial x^i} = 0, \quad \forall A.$$

Another possibility is when the matrix $(W_{AB}) = \left(\frac{\partial^2 L}{\partial y^A \partial y^B} \right)$ is singular. This type of Lagrangian is called *singular* or *degenerate Lagrangian*. In such a case, ω_L is not a symplectic section and Equation (18) has no solution, in general, and even if it exists it will not be unique. In the next subsection, we will give the extension of the classical Gotay-Nester-Hinds algorithm [37] for presymplectic systems on Lie algebroids given in [40], which in particular will be applied to optimal control problems.

For an arbitrary Lagrangian function $L : E \rightarrow \mathbb{R}$, we introduce the *Legendre transformation* associated with L as the smooth map $leg_L : E \rightarrow E^*$ defined by

$$leg_L(e)(e') = \left. \frac{d}{dt} \right|_{t=0} L(e + te'),$$

for $e, e' \in E_x$. Its local expression is

$$leg_L(x^i, y^A) = (x^i, \frac{\partial L}{\partial y^A}). \quad (19)$$

The Legendre transformation induces a Lie algebroid morphism

$$\mathcal{T}leg_L : \mathcal{T}^E E \rightarrow \mathcal{T}^{E^*} E$$

over $leg_L : E \rightarrow E^*$ given by

$$(\mathcal{T}leg_L)(e, v) = (e, (Tleg_L)(v)),$$

where $Tleg_L : TE \rightarrow TE^*$ is the tangent map of $leg_L : E \rightarrow E^*$.

We have that (see [49] for details)

$$(\mathcal{T}leg_L, leg_L)^*(\lambda_E) = \Theta_L, \quad (\mathcal{T}leg_L, leg_L)^*(\Omega_E) = \omega_L. \quad (20)$$

where λ_E is the Liouville section introduced in (11) and Ω_E is the canonical symplectic section on $\mathcal{T}^{E^*} E$.

On the other hand, from (19), it follows that the Lagrangian function L is regular if and only if $leg_L : E \rightarrow E^*$ is a local diffeomorphism.

Next, we will assume that L is *hyperregular*, that is, $leg_L : E \rightarrow E^*$ is a global diffeomorphism. Then, the pair $(\mathcal{T}leg_L, leg_L)$ is a Lie algebroid isomorphism. Moreover, we may consider the *Hamiltonian function* $H : E^* \rightarrow \mathbb{R}$ defined by

$$H = E_L \circ leg_L^{-1}$$

and the *Hamiltonian section* $\xi_H \in \Gamma(\mathcal{T}^{E^*} E)$ which is characterized by the condition

$$i_{\xi_H} \Omega_E = d^{\mathcal{T}^{E^*} E} H.$$

The integral curves of the vector field $\rho_1(\xi_H)$ on E^* satisfy the *Hamilton equations* for H

$$\frac{dx^i}{dt} = \rho_A^i \frac{\partial H}{\partial p_A}, \quad \frac{dp_A}{dt} = -\rho_A^i \frac{\partial H}{\partial x^i} - p_C C_{AB}^C \frac{\partial H}{\partial p_B}.$$

for $i \in \{1, \dots, m\}$ and $A \in \{1, \dots, n\}$ (see [49]).

In addition, the Euler-Lagrange section ξ_L associated with L and the Hamiltonian section ξ_H are $(\mathcal{T}leg_L, leg_L)$ -related, that is,

$$\xi_H \circ leg_L = \mathcal{T}leg_L \circ \xi_L.$$

Thus, if $\gamma : I \rightarrow E$ is a solution of the Euler-Lagrange equations associated with L , then $\mu = leg_L \circ \gamma : I \rightarrow E^*$ is a solution of the Hamilton equations for H and, conversely, if $\mu : I \rightarrow E^*$ is a solution of the Hamilton equations for H then $\gamma = leg_L^{-1} \circ \mu$ is a solution of the Euler-Lagrange equations for L (for more details, see [49]).

Example 23. Consider the Lie algebroid $E = TQ$, the fiber bundle of a manifold Q of dimension m . If (x^i) are local coordinates on Q , then $\left\{ \frac{\partial}{\partial x^i} \right\}$ is a local basis of $\mathfrak{X}(Q)$ and we have fiber local coordinates (x^i, \dot{x}^i) on TQ . The corresponding structure functions are $C_{ij}^k = 0$ and $\rho_i^j = \delta_i^j$ for $i, j, k \in \{1, \dots, m\}$. Therefore given a Lagrangian function $L : TQ \rightarrow \mathbb{R}$ the Euler-Lagrange equations associated to L are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) = \frac{\partial L}{\partial x^i}, \quad i = 1, \dots, m.$$

Moreover, given a Hamiltonian function $H : T^*Q \rightarrow \mathbb{R}$, the Hamilton equations associated to H are

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, \dots, m$$

where (x^i, p_i) are local coordinates on T^*Q induced by the local coordinates (x^i) and the local basis $\{dx^i\}$ of T^*Q (see [5] for example).

Example 24. Consider as a Lie algebroid the finite dimensional Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ over a point. If e_A is a basis of \mathfrak{g} and \tilde{C}_{AB}^C are the structure constants of the Lie algebra, the structures constant of the Lie algebroid \mathfrak{g} with respect to the basis $\{e_A\}$ are $C_{AB}^C = \tilde{C}_{AB}^C$ and $\rho_A^i = 0$. Denote by (y^A) and (μ_A) the local coordinates on \mathfrak{g} and \mathfrak{g}^* respectively, induced by the basis $\{e_A\}$ and its dual basis $\{e^A\}$ respectively. Given a Lagrangian function $L : \mathfrak{g} \rightarrow \mathbb{R}$ then the Euler-Lagrange equations for L are just the Euler-Poincaré equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) = \frac{\partial L}{\partial y^C} C_{AB}^C y^B.$$

Given a Hamiltonian function $H : \mathfrak{g}^* \rightarrow \mathbb{R}$ the Hamilton equations on \mathfrak{g}^* read as the Lie-Poisson equations for H

$$\dot{\mu} = ad_{\frac{\partial H}{\partial \mu}}^* \mu$$

(see [5] for example).

Example 25. Let G be a Lie group and assume that G acts free and properly on M . We denote by $\pi : M \rightarrow \widehat{M} = M/G$ the associated principal bundle. The tangent lift of the action gives a free and proper action of G on TM and $\widehat{TM} = TM/G$ is a quotient manifold. Then we consider the Atiyah algebroid \widehat{TM} over \widehat{M} .

According to Example 4, the basis $\{\hat{e}_i, \hat{e}_B\}$ induce local coordinates (x^i, y^i, \bar{y}^B) on \widehat{TM} . Given a Lagrangian function $\ell : \widehat{TM} \rightarrow \mathbb{R}$ on the Atiyah algebroid $\widehat{TM} \rightarrow \widehat{M}$, the Euler-Lagrange equations for ℓ are

$$\begin{aligned} \frac{\partial \ell}{\partial x^j} - \frac{d}{dt} \left(\frac{\partial \ell}{\partial y^j} \right) &= \frac{\partial \ell}{\partial \bar{y}^A} \left(\mathcal{B}_{ij}^A y^i + c_{DB}^A \mathcal{A}_j^B \bar{y}^B \right) \quad \forall j, \\ \frac{d}{dt} \left(\frac{\partial \ell}{\partial \bar{y}^B} \right) &= \frac{\partial \ell}{\partial \bar{y}^A} \left(C_{DB}^A \bar{y}^D - c_{DB}^A \mathcal{A}_i^D y^i \right) \quad \forall B, \end{aligned}$$

which are the Lagrange-Poincaré equations associated to a G -invariant Lagrangian $L : TM \rightarrow \mathbb{R}$ (see [17] and [49] for example) where c_{AB}^C are the structure constants of the Lie algebra according to Example 4.

3.2. Constraint algorithm for presymplectic Lie algebroids. In this section we introduce the constraint algorithm for presymplectic Lie algebroids given in [40] which generalizes the well-known Gotay-Nester-Hinds algorithm [37]. First we give a review of the Gotay-Nester-Hinds algorithm and then we introduce the construction given in [40] to the case of Lie algebroids.

3.2.1. The Gotay-Nester-Hinds algorithm of constraints. In this subsection we will briefly review the constraint algorithm of constraints for presymplectic systems (see [36] and [37]).

Take the following triple (M, Ω, H) consisting of a smooth manifold M , a closed 2-form Ω and a differentiable function $H : M \rightarrow \mathbb{R}$. On M we consider the equation

$$i_X \Omega = dH. \quad (21)$$

Since we are not assuming that Ω is nondegenerate (that is, Ω is not, in general, symplectic) then Equation (21) has no solution in general, or the solutions are not defined everywhere. In the most favorable case, Equation (21) admits a global (but not necessarily unique) solution X . In this case, we say that the system admits global dynamics. Otherwise, we select the subset of points of M , where such a solution exists. We denote by M_2 this subset and we will assume that it is a submanifold of $M = M_1$. Then the equations (21) admit a solution X defined at all points of M_2 , but X need not be tangent to M_2 , hence, does not necessarily induce a dynamics on M_2 . So we impose an additional tangency condition, and we obtain a new submanifold M_3 along which there exists a solution X , but, however,

such X needs to be tangent to M_3 . Continuing this process, we obtain a sequence of submanifolds

$$\cdots M_s \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 = M$$

where the general description of M_{l+1} is

$$M_{l+1} = \{p \in M_l \text{ such that there exists } X_p \in T_p M_l \text{ satisfying } i_{X_p} \Omega(p) = dH(p)\}.$$

If the algorithm ends at a final constraint submanifold, in the sense that at some $s \geq 1$ we have $M_{s+1} = M_s$. We will denote this final constraint submanifold by M_f . It may still happen that $\dim M_f = 0$, that is, M_f is a discrete set of points, and in this case the system does not admit a proper dynamics. But, in the case when $\dim M_f > 0$, there exists a well-defined solution X of (21) along M_f .

There is another characterization of the submanifolds M_l that we will use in the sequel. If N is a submanifold of M then we define

$$TN^\perp = \{Z \in T_p M, p \in N \text{ such that } \Omega(X, Z) = 0 \text{ for all } X \in T_p N\}.$$

Then, at any point $p \in M_l$ there exists $X_p \in T_p M_l$ verifying $i_X \Omega(p) = dH(p)$ if and only if $\langle TM_l^\perp, dH \rangle = 0$ (see [36, 37]). Hence, we can define the $l+1$ step of the constraint algorithm as

$$M_{l+1} := \{p \in M_l \text{ such that } \langle TM_l^\perp, dH \rangle(p) = 0\}.$$

3.2.2. Constraint algorithm for presymplectic Lie algebroids. Let $\tau_E : E \rightarrow M$ be a Lie algebroid and suppose that $\Omega \in \Gamma(\wedge^2 E^*)$. Then, we can define the vector bundle morphism $b_\Omega : E \rightarrow E^*$ (over the identity of M) as follows

$$b_\Omega(e) = i(e)\Omega(x), \quad \text{for } e \in E_x.$$

Now, if $x \in M$ and F_x is a subspace of E_x , we may introduce the vector subspace F_x^\perp of E_x given by

$$F_x^\perp = \{e \in E_x \mid \Omega(x)(e, f) = 0, \forall f \in F_x\}.$$

On the other hand, if $b_{\Omega_x} = b_{\Omega|_{E_x}}$ it is easy to prove that

$$b_{\Omega_x}(F_x) \subseteq (F_x^\perp)^0, \quad (22)$$

where $(F_x^\perp)^0$ is the annihilator of the subspace F_x^\perp . Moreover, using

$$\dim F_x^\perp = \dim E_x - \dim F_x + \dim(E_x^\perp \cap F_x). \quad (23)$$

we obtain that

$$\dim(F_x^\perp)^0 = \dim F_x - \dim(E_x^\perp \cap F_x) = \dim(b_{\Omega_x}(E_x)).$$

Thus, from (22), we deduce that

$$b_{\Omega_x}(F_x) = (F_x^\perp)^\circ. \quad (24)$$

Next, we will assume that Ω is a presymplectic 2-section ($d^E \Omega = 0$) and that $\alpha \in \Gamma(E^*)$ is a closed 1-section ($d^E \alpha = 0$). Furthermore, we will assume that the kernel of Ω is a vector subbundle of E .

The dynamics of the presymplectic system defined by (Ω, α) is given by a section $X \in \Gamma(E)$ satisfying the dynamical equation

$$i_X \Omega = \alpha. \quad (25)$$

In general, a section X satisfying (25) cannot be found in all points of E . First, we look for the points where (25) has sense. We define

$$M_1 = \{x \in M \mid \exists e \in E_x : i(e)\Omega(x) = \alpha(x)\}$$

From (24), it follows that

$$M_1 = \{x \in M \mid \alpha(x)(e) = 0, \text{ for all } e \in \ker \Omega(x) = E_x^\perp\}. \quad (26)$$

If M_1 is an embedded submanifold of M , then we deduce that there exists $X : M_1 \rightarrow E$ a section of $\tau_E : E \rightarrow M$ along M_1 such that (25) holds. But $\rho(X)$ is not, in general,

tangent to M_1 . Thus, we have to restrict to $E_1 = \rho^{-1}(TM_1)$. We remark that, provided that E_1 is a manifold and $\tau_1 = \tau_E|_{E_1} : E_1 \rightarrow M_1$ is a vector bundle, $\tau_1 : E_1 \rightarrow M_1$ is a Lie subalgebroid of $E \rightarrow M$.

Now, we must consider the subset M_2 of M_1 defined by

$$\begin{aligned} M_2 &= \{x \in M_1 \mid \alpha(x) \in \mathfrak{b}_{\Omega_x}((E_1)_x) = \mathfrak{b}_{\Omega_x}(\rho^{-1}(T_x M_1))\} \\ &= \{x \in M_1 \mid \alpha(x)(e) = 0, \text{ for all } e \in (E_1)_x^\perp = (\rho^{-1}(T_x M_1))^\perp\}. \end{aligned}$$

If M_2 is an embedded submanifold of M_1 , then we deduce that there exists $X : M_2 \rightarrow E_1$ a section of $\tau_1 : E_1 \rightarrow M_1$ along M_2 such that (25) holds. However, $\rho(X)$ is not, in general, tangent to M_2 . Therefore, we have that to restrict to $E_2 = \rho^{-1}(TM_2)$. As above, if $\tau_2 = \tau_E|_{E_2} : E_2 \rightarrow M_2$ is a vector bundle, it follows that $\tau_2 : E_2 \rightarrow M_2$ is a Lie subalgebroid of $\tau_1 : E_1 \rightarrow M_1$.

Consequently, if we repeat the process, we obtain a sequence of Lie subalgebroids (by assumption)

$$\begin{array}{ccccccccccc} \dots & \hookrightarrow & M_{k+1} & \hookrightarrow & M_k & \hookrightarrow \dots \hookrightarrow & M_2 & \hookrightarrow & M_1 & \hookrightarrow & M_0 = M \\ & & \uparrow \tau_{k+1} & & \uparrow \tau_k & & \uparrow \tau_2 & & \uparrow \tau_1 & & \uparrow \tau_E \\ \dots & \hookrightarrow & E_{k+1} & \hookrightarrow & E_k & \hookrightarrow \dots \hookrightarrow & E_2 & \hookrightarrow & E_1 & \hookrightarrow & E_0 = E \end{array}$$

where

$$M_{k+1} = \{x \in M_k \mid \alpha(x)(e) = 0, \text{ for all } e \in (\rho^{-1}(T_x M_k))^\perp\} \quad (27)$$

and

$$E_{k+1} = \rho^{-1}(TM_{k+1}).$$

If there exists $k \in \mathbb{N}$ such that $M_k = M_{k+1}$, then we say that the sequence stabilizes. In such a case, there exists a well-defined (but non necessarily unique) dynamics on the final constraint submanifold $M_f = M_k$. We write

$$M_f = M_{k+1} = M_k, \quad E_f = E_{k+1} = E_k = \rho^{-1}(TM_k).$$

Then, $\tau_f = \tau_k : E_f = E_k \rightarrow M_f = M_k$ is a Lie subalgebroid of $\tau_E : E \rightarrow M$ (the Lie algebroid restriction of E to E_f). From the construction of the constraint algorithm, we deduce that there exists a section $X \in \Gamma(E_f)$, verifying (25). Moreover, if $X \in \Gamma(E_f)$ is a solution of the equation (25), then every arbitrary solution is of the form $X' = X + Y$, where $Y \in \Gamma(E_f)$ and $Y(x) \in \ker \Omega(x)$, for all $x \in M_f$. In addition, if we denote by Ω_f and α_f the restriction of Ω and α , respectively, to the Lie algebroid $E_f \rightarrow M_f$, we have that Ω_f is a presymplectic 2-section and then any $X \in \Gamma(E_f)$ verifying Equation (25) also satisfies

$$i_X \Omega_f = \alpha_f \quad (28)$$

but, in principle, there are solutions of (28) which are not solutions of (25) since $\ker \Omega \cap E_f \subset \ker \Omega_f$.

Remark 1. Note that one can generalize the previous procedure to the general setting of *implicit differential equations on a Lie algebroid*. More precisely, let $\tau_E : E \rightarrow M$ be a Lie algebroid and $S \subset E$ be a submanifold of E (not necessarily a vector subbundle). Then, the corresponding sequence of submanifolds of E is

$$\begin{aligned} S_0 &= S \\ S_1 &= S_0 \cap \rho^{-1}(T\tau_E(S_0)) \\ &\vdots \\ S_{k+1} &= S_k \cap \rho^{-1}(T\tau_E(S_k)) \\ &\vdots \end{aligned}$$

In our case, $S_k = \rho^{-1}(TM_k)$ (equivalently, $M_k = \tau_E(S_k)$). \diamond

3.3. Vakonomic mechanics on Lie algebroids. In this section we will develop a geometrical description for second-order mechanics on Lie algebroids in the Skinner and Rusk formalism, given a general geometric framework for the previous results in this chapter and using strongly the results given in [40].

First, we will review the description of vakonomics mechanics on Lie algebroids given by Iglesias, Marrero, Martín de Diego and Sosa in [40]. After it we will introduce the notion of admissible elements on a Lie algebroid and we will particularize the previous construction to the case when the Lie algebroid is the prolongation of a Lie algebroid and the constraint submanifold is the set of admissible elements. Then we will obtain the second-order Skinner and Rusk formulation on Lie algebroids.

Let $\tau_{\tilde{E}} : \tilde{E} \rightarrow Q$ be a Lie algebroid of rank n over a manifold Q of dimension m with anchor map $\rho : \tilde{E} \rightarrow TQ$ and $L : \tilde{E} \rightarrow \mathbb{R}$ be a Lagrangian function on \tilde{E} . Moreover, let $\mathcal{M} \subset \tilde{E}$ be an embedded submanifold of dimension $n + m - \tilde{m}$ such that $\tau_{\mathcal{M}} = \tau_{\tilde{E}}|_{\mathcal{M}} : \mathcal{M} \rightarrow Q$ is a surjective submersion.

Suppose that e is a point of \mathcal{M} with $\tau_{\mathcal{M}}(e) = x \in Q$, (x^i) are local coordinates on an open subset U of Q , $x \in U$, and $\{e_A\}$ is a local basis of $\Gamma(\tilde{E})$ on U . Denote by (x^i, y^A) the corresponding local coordinates for \tilde{E} on the open subset $\tau_{\tilde{E}}^{-1}(U)$. Assume that

$$\mathcal{M} \cap \tau_{\tilde{E}}^{-1}(U) \equiv \{(x^i, y^A) \in \tau_{\tilde{E}}^{-1}(U) \mid \Phi^\alpha(x^i, y^A) = 0, \alpha = 1, \dots, \tilde{m}\}$$

where Φ^α are the local independent constraint functions for the submanifold \mathcal{M} .

We will suppose, without loss of generality, that the $(\tilde{m} \times n)$ -matrix

$$\left(\frac{\partial \Phi^\alpha}{\partial y^B} \right) \Big|_e \Big|_{\alpha=1, \dots, \tilde{m}; B=1, \dots, n}$$

is of maximal rank.

Now, using the implicit function theorem, we obtain that there exists an open subset \tilde{V} of $(\tau_{\tilde{E}})^{-1}(U)$, an open subset $W \subseteq \mathbb{R}^{m+n-\tilde{m}}$ and smooth real functions $\Psi^\alpha : W \rightarrow \mathbb{R}$, $\alpha = 1, \dots, \tilde{m}$, such that

$$\mathcal{M} \cap \tilde{V} \equiv \{(x^i, y^A) \in \tilde{V} \mid y^\alpha = \Psi^\alpha(x^i, y^a), \text{ with } \alpha = 1, \dots, \tilde{m} \text{ and } \tilde{m} + 1 \leq a \leq n\}.$$

Consequently, (x^i, y^a) are local coordinates on \mathcal{M} and we will denote by \tilde{L} the restriction of L to \mathcal{M} .

Consider the Whitney sum of \tilde{E}^* and \tilde{E} , that is, $W = \tilde{E} \oplus \tilde{E}^*$, and the canonical projections $pr_1 : \tilde{E} \oplus \tilde{E}^* \rightarrow \tilde{E}$ and $pr_2 : \tilde{E} \oplus \tilde{E}^* \rightarrow \tilde{E}^*$. Now, let W_0 be the submanifold $W_0 = pr_1^{-1}(\mathcal{M}) = \mathcal{M} \times_Q \tilde{E}^*$ and the restrictions $\pi_1 = pr_1|_{W_0}$ and $\pi_2 = pr_2|_{W_0}$. Also denote by $\nu : W_0 \rightarrow Q$ the canonical projection of W_0 over the base manifold.

Next, we consider the prolongation of the Lie algebroid \tilde{E} over $\tau_{\tilde{E}^*} : \tilde{E}^* \rightarrow Q$ (respectively, $\nu : W_0 \rightarrow Q$). We will denote this Lie algebroid by $\mathcal{T}^{\tau_{\tilde{E}^*}} \tilde{E}$ (respectively, $\mathcal{T}^\nu \tilde{E}$). Moreover, we can prolong $\pi_2 : W_0 \rightarrow \tilde{E}^*$ to a morphism of Lie algebroids $\mathcal{T}\pi_2 : \mathcal{T}^\nu \tilde{E} \rightarrow \mathcal{T}^{\tau_{\tilde{E}^*}} \tilde{E}$ defined by $\mathcal{T}\pi_2 = (Id, T\pi_2)$.

If (x^i, p_A) are the local coordinates on \tilde{E}^* associated with the local basis $\{e^A\}$ of $\Gamma(\tilde{E}^*)$, then (x^i, p_A, y^a) are local coordinates on W_0 and we may consider the local basis $\{\tilde{e}_A^{(1)}, (\tilde{e}^A)^{(2)}, e_a^{(2)}\}$ of $\Gamma(\mathcal{T}^\nu \tilde{E})$ defined by

$$\begin{aligned} \tilde{e}_A^{(1)}(\check{e}, e^*) &= \left(e_A(x), \rho_A^i \frac{\partial}{\partial x^i} \Big|_{(\check{e}, e^*)} \right), & (\tilde{e}^A)^{(2)}(\check{e}, e^*) &= \left(0, \frac{\partial}{\partial p_A} \Big|_{(\check{e}, e^*)} \right), \\ e_a^{(2)}(\check{e}, e^*) &= \left(0, \frac{\partial}{\partial y^a} \Big|_{(\check{e}, e^*)} \right), \end{aligned}$$

where $(\check{e}, e^*) \in W_0$ and $\nu(\check{e}, e^*) = x$. If $(\llbracket \cdot, \cdot \rrbracket^\nu, \rho^\nu)$ is the Lie algebroid structure on $\mathcal{T}^\nu \tilde{E}$, we have that

$$\llbracket \tilde{e}_A^{(1)}, \tilde{e}_B^{(1)} \rrbracket^\nu = \mathcal{C}_{AB}^C \tilde{e}_C^{(1)},$$

and the rest of the fundamental Lie brackets are zero. Moreover,

$$\rho^\nu(\tilde{e}_A^{(1)}) = \rho_A^i \frac{\partial}{\partial x^i}, \quad \rho^\nu((\tilde{e}^A)^{(2)}) = \frac{\partial}{\partial p_A}, \quad \rho^\nu(e_a^{(2)}) = \frac{\partial}{\partial y^a}.$$

The Pontryagin Hamiltonian H_{W_0} is a function defined on $W_0 = \mathcal{M} \times_Q \tilde{E}^*$ given by

$$H_{W_0}(\tilde{e}, e^*) = \langle e^*, \tilde{e} \rangle - \tilde{L}(\tilde{e}),$$

or, in local coordinates,

$$H_{W_0}(x^i, p_A, y^a) = p_A y^a + p_\alpha \Psi^\alpha(x^i, y^a) - \tilde{L}(x^i, y^a). \quad (29)$$

Moreover, one can consider the presymplectic 2-section $\Omega_0 = (\mathcal{T}\pi_2, \pi_2)^* \Omega_{\tilde{E}}$, where $\Omega_{\tilde{E}}$ is the canonical symplectic section on $\mathcal{T}^{\tau \tilde{E}^*} \tilde{E}$ defined in Equation (12). In local coordinates,

$$\Omega_0 = \tilde{e}_{(1)}^A \wedge \tilde{e}_A^{(2)} + \frac{1}{2} \mathcal{C}_{ABPC}^C \tilde{e}_{(1)}^A \wedge \tilde{e}_{(1)}^B, \quad (30)$$

where $\{\tilde{e}_{(1)}^A, \tilde{e}_A^{(2)}, e_a^{(2)}\}$ denotes the dual basis of $\{\tilde{e}_A^{(1)}, (\tilde{e}^A)^{(2)}, e_a^{(2)}\}$.

Therefore, we have the triple $(\mathcal{T}^\nu \tilde{E}, \Omega_0, d^{\mathcal{T}^\nu \tilde{E}} H_{W_0})$ as a presymplectic hamiltonian system.

Definition 3.1. The vakonomic problem on Lie algebroids consists on finding the solutions for the equation

$$i_X \Omega_0 = d^{\mathcal{T}^\nu \tilde{E}} H_{W_0}; \quad (31)$$

that is, to solve the constraint algorithm for $(\mathcal{T}^\nu \tilde{E}, \Omega_0, d^{\mathcal{T}^\nu \tilde{E}} H_{W_0})$.

In local coordinates, we have that

$$d^{\mathcal{T}^\nu \tilde{E}} H_{W_0} = \left(p_\alpha \frac{\partial \Psi^\alpha}{\partial x^i} - \frac{\partial \tilde{L}}{\partial x^i} \right) \rho_A^i \tilde{e}_{(1)}^A + \Psi^\alpha \tilde{e}_\alpha^{(2)} + y^a \tilde{e}_a^{(2)} + \left(p_a + p_\alpha \frac{\partial \Psi^\alpha}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a} \right) e_a^{(2)}.$$

If we apply the constraint algorithm,

$$W_1 = \{w \in \mathcal{M} \times_Q \tilde{E}^* \mid d^{\mathcal{T}^\nu \tilde{E}} H_{W_0}(w)(Y) = 0, \quad \forall Y \in \ker \Omega_0(w)\}.$$

Since $\ker \Omega_0 = \text{span} \{e_a^{(2)}\}$, we get that W_1 is locally characterized by the equations

$$\varphi_a = d^{\mathcal{T}^\nu \tilde{E}} H_{W_0}(e_a^{(2)}) = p_a + p_\alpha \frac{\partial \Psi^\alpha}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a} = 0,$$

or

$$p_a = \frac{\partial \tilde{L}}{\partial y^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial y^a}, \quad \bar{m} + 1 \leq a \leq n.$$

Let us also look for the expression of X satisfying Eq. (31). A direct computation shows that

$$X = y^a \tilde{e}_a^{(1)} + \Psi^\alpha \tilde{e}_\alpha^{(1)} + \left[\left(\frac{\partial \tilde{L}}{\partial x^i} - p_\alpha \frac{\partial \Psi^\alpha}{\partial x^i} \right) \rho_A^i - y^a \mathcal{C}_{Aa}^B p_B - \Psi^\alpha \mathcal{C}_{A\alpha}^B p_B \right] (\tilde{e}^A)^{(2)} + \Upsilon^a e_a^{(2)}.$$

Therefore, the vakonomic equations are

$$\begin{cases} \dot{x}^i = y^a \rho_a^i + \Psi^\alpha \rho_\alpha^i, \\ \dot{p}_\alpha = \left(\frac{\partial \tilde{L}}{\partial x^i} - p_\beta \frac{\partial \Psi^\beta}{\partial x^i} \right) \rho_\alpha^i - y^a \mathcal{C}_{\alpha a}^B p_B - \Psi^\beta \mathcal{C}_{\alpha \beta}^B p_B, \\ \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial y^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial y^a} \right) = \left(\frac{\partial \tilde{L}}{\partial x^i} - p_\alpha \frac{\partial \Psi^\alpha}{\partial x^i} \right) \rho_a^i - y^b \mathcal{C}_{ab}^B p_B - \Psi^\alpha \mathcal{C}_{a\alpha}^B p_B. \end{cases}$$

Of course, we know that there exist sections X of $\mathcal{T}^\nu \tilde{E}$ along W_1 satisfying (31), but they may not be sections of $(\rho^\nu)^{-1}(TW_1) = \mathcal{T}^{\nu_1} \tilde{E}$, in general (here $\nu_1 : W_1 \rightarrow Q$).

Then, following the procedure detailed in Section 3.2.2, we obtain a sequence of embedded submanifolds

$$\dots \hookrightarrow W_{k+1} \hookrightarrow W_k \hookrightarrow \dots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = \mathcal{M} \times_Q \tilde{E}^*.$$

If the algorithm stabilizes, then we find a final constraint submanifold W_f on which at least a section $X \in \Gamma(\mathcal{T}^{\nu_f} E)$ verifies

$$(i_X \Omega_0 = d^{\mathcal{T}^{\nu} \tilde{E}} H_{W_0})|_{W_f}$$

where $\nu_f : W_f \rightarrow Q$.

One of the most important cases is when $W_f = W_1$. The authors of [40] have analyzed this case with the following result: Consider the restriction Ω_1 of Ω_0 to $\mathcal{T}^{\nu_1} \tilde{E}$;

Proposition 2. *Ω_1 is a symplectic section of the Lie algebroid $\mathcal{T}^{\nu_1} \tilde{E}$ if and only if for any system of coordinates (x^i, p_A, y^a) on W_0 we have that*

$$\det \left(\frac{\partial^2 \tilde{L}}{\partial y^a \partial y^b} - p_\alpha \frac{\partial^2 \Psi^\alpha}{\partial y^a \partial y^b} \right) \neq 0, \text{ for all point in } W_1.$$

3.4. Second-order variational problems on Lie algebroids. In this section we will study second-order variational problems on Lie algebroid. First we introduce the geometric object for the formalism and then we study second-order unconstrained variation problems. After that, we will analyze the constrained case.

3.4.1. Prolongation of a Lie algebroid over a smooth map (cont'd). This subsection is devoted to study some additional properties and characterizations about the prolongation of a Lie algebroid over a smooth map (see subsection 2.2).

Let \tilde{E} be a Lie algebroid over Q with fiber bundle projection $\tau_{\tilde{E}} : \tilde{E} \rightarrow Q$ and anchor map $\rho : \tilde{E} \rightarrow TQ$. Also, let $\tau_E : E \rightarrow M$ be a Lie algebroid with anchor map $\rho : E \rightarrow TM$ and let $\mathcal{T}^{\tau_E} E$ be the E -tangent bundle to E . Now we will define the bundle $\mathcal{T}^{\tau_E^{(1)}}(\mathcal{T}^{\tau_E} E)$ over $\mathcal{T}^{\tau_E} E$. This bundle plays the role of $\tau_{T(TM)} : T(TTM) \rightarrow T(TM)$ in ordinary Lagrangian Mechanics.

In what follows we will describe the Lie algebroid structure of the E -tangent bundle to the prolongation Lie algebroid over $\tau_E : E \rightarrow Q$.

As we know from subsection (2.2), the basis of sections $\{e_A\}$ of E induces a local basis of the sections of $\mathcal{T}^{\tau_E} E$ given by

$$e_A^{(1)}(e) = \left(e, e_A(\tau_E(e)), \rho_A^i \frac{\partial}{\partial x^i} \Big|_e \right), \quad e_A^{(2)}(e) = \left(e, 0, \frac{\partial}{\partial y^A} \Big|_e \right),$$

for $e \in E$. From this basis we can induce local coordinates $(x^i, y^A; z^A, v^A)$ on $\mathcal{T}^{\tau_E} E$. Now, from this basis, we can induce a local basis of sections of $\mathcal{T}^{\tau_E^{(1)}}(\mathcal{T}^{\tau_E} E)$ in the following way: consider an element $(e, v_b) \in \mathcal{T}^{\tau_E} E$, then define the components of the basis $\{e_A^{(1,1)}, e_A^{(2,1)}, e_A^{(1,2)}, e_A^{(2,2)}\}$ as

$$\begin{aligned} e_A^{(1,1)}(e, v_b) &= \left((e, v_b), e_A^{(1)}(e), \rho_A^i \frac{\partial}{\partial x^i} \Big|_{(e, v_b)} \right), & e_A^{(2,1)}(e, v_b) &= \left((e, v_b), e_A^{(2)}(e), \frac{\partial}{\partial y^A} \Big|_{(e, v_b)} \right), \\ e_A^{(1,2)}(e, v_b) &= \left((e, v_b), 0, \frac{\partial}{\partial z^A} \Big|_{(e, v_b)} \right), & e_A^{(2,2)}(e, v_b) &= \left((e, v_b), 0, \frac{\partial}{\partial v^A} \Big|_{(e, v_b)} \right). \end{aligned}$$

The basis $\{e_A^{(1,1)}, e_A^{(2,1)}, e_A^{(1,2)}, e_A^{(2,2)}\}$ induces local coordinates $(x^i, y^A, z^A, v^A, b^A, c^A, d^A, w^A)$ on $\mathcal{T}^{\tau_E^{(1)}}(\mathcal{T}^{\tau_E} E)$. If we denote by $(\mathcal{T}^{\tau_E^{(1)}}(\mathcal{T}^{\tau_E} E), \llbracket \cdot, \cdot \rrbracket_{\tau_E^{(2)}}, \rho_2)$ the Lie algebroid structure

Consider the prolongations of $\mathcal{T}^{\tau E} E$ by $\tau_{(\mathcal{T}^{\tau E} E)^*}$ and by ν , respectively. We will denote these Lie algebroids by $\mathcal{T}^{\tau(\mathcal{T}^{\tau E} E)^*}(\mathcal{T}^{\tau E} E)$ and $\mathcal{T}^\nu \mathcal{T}^{\tau E} E$ respectively. Moreover, we can prolong $\pi_2 : W_0 \rightarrow (\mathcal{T}^{\tau E} E)^*$ to a morphism of Lie algebroids $\mathcal{T}\pi_2 : \mathcal{T}^\nu \mathcal{T}^{\tau E} E \rightarrow \mathcal{T}^{\tau(\mathcal{T}^{\tau E} E)^*}(\mathcal{T}^{\tau E} E)$ defined by $\mathcal{T}\pi_2 = (Id, T\pi_2)$.

We denote by $(x^i, y^A, p_A, \bar{p}_A)$ local coordinates on $(\mathcal{T}^{\tau E} E)^*$ induced by $\{e_{(1)}^A, e_{(2)}^A\}$, the dual basis of the basis $\{e_A^{(1)}, e_A^{(2)}\}$, a basis of $\mathcal{T}^{\tau E} E$. Then, $(x^i, y^A, p_A, \bar{p}_A, z^A)$ are local coordinates in W_0 and we may consider $\{\tilde{e}_A^{(1,1)}, \tilde{e}_A^{(2,1)}, (\tilde{e}^A)^{(1,2)}, (\tilde{e}^A)^{(2,2)}, \tilde{e}_A^{(1,2)}\}$, the local basis of $\Gamma(\mathcal{T}^\nu \mathcal{T}^{\tau E} E)$ defined as

$$\begin{aligned} \tilde{e}_A^{(1,1)}(\check{\alpha}, \alpha^*) &= \left((\check{\alpha}, \alpha^*), e_A^{(1)}(\tau_{(\mathcal{T}^{\tau E} E)^*}(\alpha^*)), \rho_A^i \frac{\partial}{\partial x^i} \Big|_{(\check{\alpha}, \alpha^*)} \right), \\ (\tilde{e}^A)^{(1,2)}(\check{\alpha}, \alpha^*) &= \left((\check{\alpha}, \alpha^*), 0, \frac{\partial}{\partial p_A} \Big|_{(\check{\alpha}, \alpha^*)} \right), \\ \tilde{e}_A^{(2,1)}(\check{\alpha}, \alpha^*) &= \left((\check{\alpha}, \alpha^*), e_A^{(2)}(\tau_{(\mathcal{T}^{\tau E} E)^*}(\alpha^*)), \frac{\partial}{\partial y^A} \Big|_{(\check{\alpha}, \alpha^*)} \right), \\ (\tilde{e}^A)^{(2,2)}(\check{\alpha}, \alpha^*) &= \left((\check{\alpha}, \alpha^*), 0, \frac{\partial}{\partial \bar{p}_A} \Big|_{(\check{\alpha}, \alpha^*)} \right), \quad \tilde{e}_A^{(1,2)}(\check{\alpha}, \alpha^*) = \left((\check{\alpha}, \alpha^*), 0, \frac{\partial}{\partial z^A} \Big|_{(\check{\alpha}, \alpha^*)} \right) \end{aligned}$$

for $\alpha^* \in (\mathcal{T}^{\tau E} E)^*$, $\check{\alpha} \in E^{(2)}$, $(\check{\alpha}, \alpha^*) \in W_0$, and $\tau_{(\mathcal{T}^{\tau E} E)^*} : (\mathcal{T}^{\tau E} E)^* \rightarrow E$ is the canonical projection.

If $([\![\cdot, \cdot]\!]^\nu, \rho^\nu)$ is the Lie algebroid structure on $\mathcal{T}^\nu \mathcal{T}^{\tau E} E$, we have that $[\![\tilde{e}_A^{(1,1)}, \tilde{e}_B^{(1,1)}]\!]^\nu = \mathcal{C}_{AB}^C \tilde{e}_C^{(1,1)}$, and the rest of the fundamental Lie brackets are zero. Moreover,

$$\begin{aligned} \rho^\nu(\tilde{e}_A^{(1,1)}(\check{\alpha}, \alpha^*)) &= \left((\check{\alpha}, \alpha^*), \rho_A^i \frac{\partial}{\partial x^i} \Big|_{(\check{\alpha}, \alpha^*)} \right), \\ \rho^\nu((\tilde{e}^A)^{(1,2)}(\check{\alpha}, \alpha^*)) &= \left((\check{\alpha}, \alpha^*), \frac{\partial}{\partial p_A} \Big|_{(\check{\alpha}, \alpha^*)} \right), \quad \rho^\nu(\tilde{e}_A^{(2,1)}(\check{\alpha}, \alpha^*)) = \left((\check{\alpha}, \alpha^*), \frac{\partial}{\partial y^A} \Big|_{(\check{\alpha}, \alpha^*)} \right), \\ \rho^\nu(\tilde{e}_A^{(2,1)}(\check{\alpha}, \alpha^*)) &= \left((\check{\alpha}, \alpha^*), \frac{\partial}{\partial y^A} \Big|_{(\check{\alpha}, \alpha^*)} \right), \quad \rho^\nu((\tilde{e}^A)^{(2,2)}(\check{\alpha}, \alpha^*)) = \left((\check{\alpha}, \alpha^*), \frac{\partial}{\partial \bar{p}_A} \Big|_{(\check{\alpha}, \alpha^*)} \right). \end{aligned}$$

The Pontryagin Hamiltonian H_{W_0} is a function in W_0 given by

$$H_{W_0}(\check{\alpha}, \alpha^*) = \langle \alpha^*, \check{\alpha} \rangle - L(\check{\alpha}),$$

or in local coordinates

$$H_{W_0}(x^i, y^A, p_A, \bar{p}_A, z^A) = \bar{p}_A z^A + p_A y^A - L(x^i, y^A, z^A).$$

Moreover, one can consider the presymplectic 2-section $\Omega_0 = (\mathcal{T}\pi_2, \pi_2)^* \Omega_E$, where Ω_E is the canonical symplectic section on $\mathcal{T}^{\tau E} E$. In local coordinates,

$$\Omega_0 = \tilde{e}_{(1,1)}^A \wedge (\tilde{e}_A)^{(1,2)} + \tilde{e}_{(2,1)}^A \wedge (\tilde{e}_A)^{(2,2)} + \frac{1}{2} \tilde{\mathcal{C}}_{AB}^C p_C \tilde{e}_{(1,1)}^A \wedge \tilde{e}_{(1,1)}^B.$$

Here, the basis $\{\tilde{e}_{(1,1)}^A, \tilde{e}_{(2,1)}^A, (\tilde{e}_A)^{(1,2)}, (\tilde{e}_A)^{(2,2)}, \tilde{e}_{(1,2)}^A\}$ denotes the dual basis of the basis of sections for $\mathcal{T}^{\tau(\mathcal{T}^{\tau E} E)^*} \mathcal{T}^{\tau E} E$, denoted by $\{\tilde{e}_A^{(1,1)}, \tilde{e}_A^{(2,1)}, (\tilde{e}^A)^{(1,2)}, (\tilde{e}^A)^{(2,2)}, \tilde{e}_A^{(1,2)}\}$.

Therefore, the triple $(\mathcal{T}^\nu \mathcal{T}^{\tau E} E, \Omega_0, d^{\mathcal{T}^\nu \mathcal{T}^{\tau E} E} H_{W_0})$ is a presymplectic Hamiltonian system.

The second-order problem on the Lie algebroid $\tau_E : E \rightarrow M$ consists on finding the solutions of the equation

$$i_X \Omega_0 = d^{\mathcal{T}^\nu \mathcal{T}^{\tau E} E} H_{W_0},$$

that is, to solve the constraint algorithm for $(\mathcal{T}^\nu \mathcal{T}^{\tau E} E, \Omega_0, d^{\mathcal{T}^\nu \mathcal{T}^{\tau E} E} H_{W_0})$.

In adapted coordinates,

$$\begin{aligned} d^{\mathcal{T}^\nu \mathcal{T}^{\tau^E} E} H_{W_0} = & -\rho_A^i \frac{\partial L}{\partial x^i} \tilde{e}_{(1,1)}^A + \left(p_A - \frac{\partial L}{\partial y^A} \right) \tilde{e}_{(2,1)}^A + \left(\bar{p}_A - \frac{\partial L}{\partial z^A} \right) \tilde{e}_{(2,1)}^A \\ & + z^A (\tilde{e}_A)^{(2,2)} + y^A (\tilde{e}_A)^{(1,2)}. \end{aligned}$$

If we apply the constraint algorithm, since $\ker \Omega_0 = \text{span} \{ \tilde{e}_A^{(2,1)} \}$ the first constraint submanifold W_1 is locally characterized by the equation

$$\varphi_A = d^{\mathcal{T}^\nu \mathcal{T}^{\tau^E} E} H_{W_0}(\tilde{e}_A^{(2,1)}) = \bar{p}_A - \frac{\partial L}{\partial z^A} = 0,$$

or

$$\bar{p}_A = \frac{\partial L}{\partial z^A}.$$

Looking for the expression of X satisfying the equation for the second-order problem we have that the second-order equations are

$$\begin{aligned} \dot{x}^i &= \rho_A^i y^A, \\ \dot{p}_A &= \rho_A^i \frac{\partial L}{\partial x^i} + C_{AB}^C p_C y^B, \\ \dot{\bar{p}}_A &= -p_A + \frac{\partial L}{\partial y^A}, \\ \bar{p}_A &= \frac{\partial L}{\partial z^A}. \end{aligned}$$

After some straightforward computations the last equations are equivalent to the following equations:

$$0 = \frac{d^2}{dt^2} \frac{\partial L}{\partial z^A} + C_{AB}^C y^B \frac{d}{dt} \left(\frac{\partial L}{\partial z^A} \right) - \frac{d}{dt} \frac{\partial L}{\partial y^A} - C_{AB}^C y^B \left(\frac{\partial L}{\partial y^A} \right) + \rho_A^i \frac{\partial L}{\partial x^i}. \quad (32)$$

As in the previous section, it is possible to apply the constraint algorithm (3.2.2) to obtain a final constraint submanifold where we have at least a solution which is dynamically compatible. The algorithm is exactly the same but applied to the equation $i_X \Omega_0 = d^{\mathcal{T}^\nu \mathcal{T}^{\tau^E} E} H_{W_0}$. Observe that the first constraint submanifold W_1 is determined by the conditions

$$\varphi_A = \bar{p}_A - \frac{\partial L}{\partial z^A} = 0.$$

If we denote by Ω_{W_1} the pullback of the presymplectic 2-section Ω_{W_0} to W_1 , then we deduce the following:

Proposition 3. Ω_{W_1} is a symplectic section of the Lie algebroid $\mathcal{T}^{\nu_1} \mathcal{T}^{\tau^E} E$ if and only if

$$\left(\frac{\partial^2 L}{\partial z^A \partial z^B} \right)$$

is nondegenerate along W_1 , where $\nu_1 = \nu|_{W_1}: W_1 \rightarrow E$.

Remark 2. Proposition 3 is the same result than the theorem given in [40] explained in section 3.3 to the particular case when the $M = E^{(2)}$. \diamond

Example 26. Observe that we can particularize the equations (32) to the case of Atiyah algebroids to obtain the second-order Lagrange-Poincaré equations.

Let G be a Lie group and we assume that G acts free and properly on M . We denote by $\pi: M \rightarrow \widehat{M} = M/G$ the associated principal bundle. The tangent lift of the action gives a free and proper action of G on TM and $\widehat{TM} = TM/G$ is a quotient manifold. Then we consider the Atiyah algebroid \widehat{TM} over \widehat{M} .

According to example 4, the basis $\{\hat{e}_i, \hat{e}_B\}$ induce local coordinates (x^i, y^i, \bar{y}^B) . From this basis one can induce a basis of the prolongation Lie algebroid, namely $\{\hat{e}_i^{(1)}, \hat{e}_B^{(1)}\}$. This basis induce adapted coordinates $(x^i, y^i, \bar{y}^B, \dot{y}^i, \dot{\bar{y}}^B)$ on $\widehat{T^{(2)}M} = (T^{(2)}M)/G$.

Given a Lagrangian function $\ell : \widehat{T^{(2)}M} \rightarrow \mathbb{R}$ over the set of admissible elements of the Atiyah algebroid $\widehat{TTM} \rightarrow \widehat{TM}$, where $\widehat{TTM} = (TTM)/G$, the Euler-Lagrange equations for ℓ are

$$\begin{aligned} \frac{\partial \ell}{\partial x^j} - \frac{d}{dt} \left(\frac{\partial \ell}{\partial \dot{y}^j} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \ell}{\partial \ddot{y}^j} \right) &= \left(\frac{d}{dt} \left(\frac{\partial \ell}{\partial \dot{y}^A} \right) - \frac{\partial \ell}{\partial \bar{y}^A} \right) \left(\mathcal{B}_{ij}^A y^i + c_{DB}^A \mathcal{A}_j^B \bar{y}^B \right) \quad \forall j, \\ \frac{d^2}{dt^2} \left(\frac{\partial \ell}{\partial \ddot{y}^B} \right) - \frac{d}{dt} \left(\frac{\partial \ell}{\partial \bar{y}^B} \right) &= \left(\frac{d}{dt} \left(\frac{\partial \ell}{\partial \dot{y}^A} \right) - \frac{\partial \ell}{\partial \bar{y}^A} \right) \left(c_{DB}^A \bar{y}^D - c_{DB}^A \mathcal{A}_i^D y^i \right), \forall B \end{aligned}$$

which are the second-order Lagrange-Poincaré equations associated to a G -invariant Lagrangian $L : T^{(2)}M \rightarrow \mathbb{R}$ (see [30] and [31]) where c_{AB}^C are the structure constants of the Lie algebra according to Example 4.

Observe that If $G = \{e\}$, the identity of G , $\widehat{T^{(2)}M} = T^{(2)}M$ and the second-order Lagrange-Poincaré equations become into the second-order Euler-Lagrange equations [19], [48]

$$0 = \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \dot{y}^A} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) + \frac{\partial L}{\partial x^i}.$$

If $G = M$, $\widehat{T^{(2)}M} = 2\mathfrak{g}$ after a left-trivialization, and the second-order Lagrange-Poincaré equations become into the second-order Euler-Poincaré equations [20], [29], [30]

$$0 = \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \dot{y}^A} \right) + c_{AB}^C y^B \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}^A} \right) - \frac{d}{dt} \frac{\partial L}{\partial y^A} - c_{AB}^C y^B \left(\frac{\partial L}{\partial y^A} \right).$$

3.4.3. Second-order constrained problem on Lie algebroids. Now, we will consider second-order mechanical systems subject to second-order constraints. Let $\mathcal{M} \subset E^{(2)}$ be an embedded submanifold of dimension $n + m - \bar{m}$ (locally determined by the vanishing of the constraint functions $\Phi^\alpha : \mathcal{M} \rightarrow \mathbb{R}$, $\alpha = 1, \dots, m$) such that the bundle projection $\tau_E^{(2,1)}|_{\mathcal{M}} : \mathcal{M} \rightarrow E$ is a surjective submersion.

We will suppose that the $(\bar{m} \times n)$ -matrix $\left(\frac{\partial \Phi^\alpha}{\partial z^B} \right)$ with $\alpha = 1, \dots, \bar{m}$ and $B = 1, \dots, n$ is of maximal rank. Then, we will use the following notation $z^A = (z^\alpha, z^a)$ for $1 \leq A \leq n$, $1 \leq \alpha \leq \bar{m}$ and $\bar{m} + 1 \leq a \leq n$. Therefore, using the implicit function theorem we can write

$$z^\alpha = \Psi^\alpha(x^i, y^A, z^a).$$

Consequently we can consider local coordinates on \mathcal{M} by (x^i, y^A, z^a) and we will denote by \tilde{L} the restriction of L to \mathcal{M} .

Proposition 4 ([51]). *Let $(E, [\cdot, \cdot], \rho)$ be a Lie algebroid over a manifold M with projection $\tau_E : E \rightarrow M$ and anchor map with constant rank. Consider a submanifold N of M . If $\tau_E|_{\rho^{-1}(TN)} : \rho^{-1}(TN) \rightarrow M$ is a vector subbundle, then $\rho^{-1}(TN)$ is a Lie algebroid over N .*

Let us take the submanifold $\overline{W}_0 = pr_1^{-1}(\mathcal{M}) = \mathcal{M} \times_E (\mathcal{T}^{\tau_E} E)^*$ and the restrictions of \overline{W}_0 of the canonical projections π_1 and π_2 given by $\pi_1 = pr_1|_{\overline{W}_0}$ and $\pi_2 = pr_1|_{\overline{W}_0}$. We will denote local coordinates on \overline{W}_0 by $(x^i, y^A, p_A, \bar{p}_A, z^a)$.

Therefore, proceeding as in the unconstrained case one can construct the presymplectic Hamiltonian system $(\overline{W}_0, \Omega_{\overline{W}_0}, H_{\overline{W}_0})$, where $\Omega_{\overline{W}_0}$ is the presymplectic 2-section on \overline{W}_0 and the Hamiltonian function $H : \overline{W}_0 \rightarrow \mathbb{R}$ is locally given by

$$H_{\overline{W}_0}(x^i, y^A, p_A, \bar{p}_A, z^a) = p_A y^A + \bar{p}_a z^a + \bar{p}_\alpha \Psi^\alpha(x^i, y^A, z^a) - \tilde{L}(x^i, y^A, z^a).$$

With these two elements it is possible to write the following presymplectic system

$$i_X \Omega_{\overline{W}_0} = d^{(\rho^\nu)^{-1}(T\overline{W}_0)} H_{\overline{W}_0}, \quad (33)$$

where $(\rho^\nu)^{-1}(T\overline{W}_0)$ denotes the Lie subalgebroid of $\mathcal{T}^\nu \mathcal{T}^{\tau_E} E$ over $\overline{W}_0 \subset W_0$.

To characterize the equations we will adopt an “extrinsic point of view”, that is, we will work on the full space W_0 instead of in the restricted space \overline{W}_0 . Consider an arbitrary

extension $L : E^{(2)} \rightarrow \mathbb{R}$ of $L_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{R}$. The main idea is to take into account that Equation (33) is equivalent to

$$\begin{cases} i_X \Omega_{W_0} - d^{T^\nu T^{\tau E} E} H & \in \text{ann } (\rho^\nu)^{-1}(T_x \overline{W}_0), \\ X & \in (\rho^\nu)^{-1}(T_x \overline{W}_0) \text{ and } x \in \overline{W}_0, \end{cases}$$

where $H : W_0 \rightarrow \mathbb{R}$ is the function defined in the last section and ann denotes the set of sections $\tilde{X} \in \Gamma((\mathcal{T}^\nu \mathcal{T}^{\tau E} E)^*)$ such that $\langle \tilde{X}, Y \rangle = 0$ for all $Y \in (\rho^\nu)^{-1}(TW_0)$.

Assuming that \mathcal{M} is determined by the vanishing of \overline{m} -independent constraints

$$\Phi^\alpha(x^i, y^A, z^a) = 0, \quad 1 \leq \alpha \leq \overline{m},$$

then, locally, $\text{ann } (\rho^\nu)^{-1}(T\overline{W}_0) = \text{span } \{d^{T^\nu T^{\tau E} E} \Phi^\alpha\}$, and therefore the previous equations are rewritten as

$$\begin{cases} i_X \Omega_{W_0} - d^{T^\nu T^{\tau E} E} H & = \lambda_\alpha d^{T^\nu T^{\tau E} E} \Phi^\alpha, \\ X(x) \in (\rho^\nu)^{-1}(T_x \overline{W}_0) & \text{for all } x \in \overline{W}_0, \end{cases}$$

where λ_α are Lagrange multipliers to be determined.

Proceeding as in the previous section, one can obtain the following system of equations for $\tilde{L} = L + \lambda_\alpha \Phi^\alpha$

$$\begin{aligned} 0 &= \frac{d^2}{dt^2} \frac{\partial \tilde{L}}{\partial z^A} + C_{AB}^C y^B \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial z^A} \right) - \frac{d}{dt} \frac{\partial \tilde{L}}{\partial y^A} - C_{AB}^C y^B \left(\frac{\partial \tilde{L}}{\partial y^A} \right) + \rho_A^i \frac{\partial \tilde{L}}{\partial x^i} \\ 0 &= \Phi^\alpha(x^i, y^A, z^A). \end{aligned} \quad (34)$$

Here the first constraint submanifold \overline{W}_1 is determined by the condition

$$\begin{aligned} 0 &= \bar{p}_A - \frac{\partial L}{\partial z^A} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial z^A} \\ 0 &= \Phi^\alpha(x^i, y^A, z^A). \end{aligned}$$

If we denote by $\Omega_{\overline{W}_1}$ the pullback of the presymplectic section $\Omega_{\overline{W}_0}$ to \overline{W}_1 , then we can deduce that $\Omega_{\overline{W}_1}$ is a symplectic section if and only if

$$\begin{pmatrix} \frac{\partial^2 L}{\partial z^A \partial z^B} + \lambda_\alpha \frac{\partial^2 \Phi^\alpha}{\partial z^A \partial z^B} & \frac{\partial \Phi^\alpha}{\partial z^A} \\ \frac{\partial \Phi^\alpha}{\partial z^B} & \mathbf{0} \end{pmatrix} \quad (35)$$

is nondegenerate.

4. APPLICATION TO OPTIMAL CONTROL OF MECHANICAL SYSTEMS

In this section we study optimal control problems of mechanical systems defined on Lie algebroids. First we treat with fully actuated system and next with underactuated systems. Optimality conditions for the optimal control of the controlled Elroy's Beany system are derived.

Optimal control problems can be seen as higher-order variational problems (see [5] and [6]). Higher-order variational problems are given by

$$\min_{q(\cdot)} \int_0^T L(q^i, \dot{q}^i, \dots, q^{(k)i}) dt,$$

subject to boundary conditions. The relationship between higher-order variational problems and optimal control problems of mechanical systems comes from the fact that Euler-Lagrange equations are represented by a second-order Newtonian system and mechanical control systems have the form $F(q^i, \dot{q}^i, \ddot{q}^i) = u$, where u are the control inputs. Then, if C is a given cost function,

$$\min_{(q(\cdot), u(\cdot))} \int_0^T C(q^i, \dot{q}^i, u) dt,$$

is equivalent to a higher-order variational problem with $k = 2$.

4.1. Optimal control problems of fully-actuated mechanical systems on Lie algebroids. Let $(E, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid over Q with bundle projection $\tau_E : E \rightarrow Q$. The dynamics is specified fixing a Lagrangian $L : E \rightarrow \mathbb{R}$. External forces are modeled, in this case, by curves $u_F : \mathbb{R} \rightarrow E^*$ where E^* is the dual bundle $\tau_{E^*} : E^* \rightarrow Q$.

Given local coordinates (q^i) on Q , and fixing a basis of sections $\{e_A\}$ of $\tau_E : E \rightarrow Q$ we can induce local coordinates (q^i, y^A) on E ; that is, every element $b \in E_q = \tau_E^{-1}(q)$ is expressed univocally as $b = y^A e_A(q)$.

It is possible to adapt the derivation of the Lagrange-d'Alembert principle to study fully-actuated mechanical controlled systems on Lie algebroids (see [26] and [59]). Let q_0 and q_T fixed in Q , consider an admissible curve $\xi : I \subset \mathbb{R} \rightarrow E$ which satisfies the principle

$$0 = \delta \int_0^T L(\xi(t)) dt + \int_0^T \langle u_F(t), \eta(t) \rangle dt,$$

where $\eta(t) \in E_{\tau_E(\xi(t))}$ and $u_F(t) \in E_{\tau_{E^*}(\xi(t))}^*$ defines the control force (where we are assuming they are arbitrary). The infinitesimal variations in the variational principle are given by $\delta \xi = \eta^C$, for all time-dependent sections $\eta \in \Gamma(\tau_E)$, with $\eta(0) = 0$ and $\eta(T) = 0$, where η^C is a time-dependent vector field on E , the *complete lift*, locally defined by

$$\eta^C = \rho_A^i \eta^A \frac{\partial}{\partial q^i} + (\dot{\eta} + C_{BC}^A \eta^B y^C) \frac{\partial}{\partial y^A}$$

(see [26, 55, 56, 57]). Here the structure functions C_{BC}^A are determined by $\llbracket e_B, e_C \rrbracket = C_{BC}^A e_A$.

From the Lagrange-d'Alembert principle one easily derives the controlled Euler-Lagrange equations by using standard variational calculus

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C(q) y^B \frac{\partial L}{\partial y^C} &= (u_F)_A, \\ \frac{dq^i}{dt} &= \rho_A^i y^A. \end{aligned}$$

where $(u_F)_A(t) = \langle u_F(t), e_A(q(t)) \rangle$ are the local components of u_F fixed the system of coordinates (q^i) on Q and the basis of section $\{e_A\}$.

The control force u_F is chosen such that it minimizes the cost functional

$$\int_0^T C(q^i, y^A, (u_F)_A) dt,$$

where $C : E \oplus E^* \rightarrow \mathbb{R}$ is the cost function associated with the optimal control problem.

Therefore, the optimal control problem consists on finding an admissible curve $\xi(t) = (q^i(t), y^A(t))$ solution of the controlled Euler-Lagrange equations, the boundary conditions and minimizing the cost functional for $C : E \oplus E^* \rightarrow \mathbb{R}$. This optimal control problem can be equivalently solved as a second-order variational problem by defining the second-order Lagrangian $\tilde{L} : E^{(2)} \rightarrow \mathbb{R}$ as

$$\tilde{L}(q^i, y^A, z^A) = C \left(q^i, y^A, \frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C(q) y^B \frac{\partial L}{\partial y^C} \right) \quad (36)$$

where we are considering local coordinates (q^i, y^A, z^A) on $E^{(2)}$.

Consider $W_0 = E^{(2)} \times (\mathcal{T}^{\tau_{E^*}} E)^*$ with local coordinates $(q^i, y^A, p_a, \bar{p}_A, z^A)$. The optimality conditions are determined by

$$\begin{aligned} \dot{q}^i &= \rho_A^i y^A, \\ \dot{p}_A &= \rho_A^i \frac{\partial C}{\partial q^i} + C_{AB}^C p_C y^B, \\ \dot{\bar{p}}_A &= -p_A + \frac{\partial C}{\partial y^A}, \\ \bar{p}_A &= \frac{\partial C}{\partial z^A}. \end{aligned}$$

The constraint submanifold W_1 is determined by $\bar{p}_A - \frac{\partial C}{\partial z^A} = 0$. If the matrix $\left(\frac{\partial^2 C}{\partial z^A \partial z^B} \right)$ is non-singular then we can write the previous equations as an explicit system of ordinary differential equations. This regularity assumption is equivalent to the condition that the constraint algorithm stops at the first constraint submanifold W_1 . Proceeding as in the previous section, after some computations, the dynamics associated with the second-order Lagrangian $\tilde{L} : E^{(2)} \rightarrow \mathbb{R}$ (and therefore the optimality conditions for the optimal control problem) is given by the second-order Euler-Lagrange equations on Lie algebroids

$$\frac{d^2}{dt^2} \left(\frac{\partial \tilde{L}}{\partial z^A} \right) + C_{AB}^C(q) y^B \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial z^C} \right) - \frac{d}{dt} \frac{\partial \tilde{L}}{\partial y^A} - C_{AB}^C(q) y^B \frac{\partial \tilde{L}}{\partial y^C} + \rho_A^i \frac{\partial \tilde{L}}{\partial q^i} = 0, \quad (37)$$

together with the admissibility condition $\frac{dq^i}{dt} = \rho_A^i y^A$.

Remark 3. Alternatively, one can define the Lagrangian $\tilde{L} : E^{(2)} \rightarrow \mathbb{R}$ in terms of the Euler-Lagrange operator as

$$\tilde{L} = C \circ (\tau_E^{E^{(2)}} \oplus \mathcal{EL}(L)) : E^{(2)} \rightarrow \mathbb{R},$$

where $\mathcal{EL}(L) : E^{(2)} \rightarrow E^*$ is the *Euler-Lagrange operator* which locally reads as

$$\mathcal{EL}(L) = \left(\frac{d}{dt} \frac{\partial L}{\partial y^A} - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^D(q) y^B \frac{\partial L}{\partial y^D} \right) e^A.$$

Here $\{e^A\}$ is the dual basis of $\{e_A\}$, the basis of sections of E and $\tau_E^{E^{(2)}} : E^{(2)} \rightarrow E$ is the canonical projection between $E^{(2)}$ and E given by the map $E^{(2)} \ni (q^i, y^A, z^A) \mapsto (q^i, y^A) \in E$. \diamond

Example 27. An illustrative example: optimal control of a fully actuated rigid body and cubic splines on Lie groups

We consider the motion of a rigid body where the configuration space is the Lie group $G = SO(3)$ and $\mathfrak{so}(3) \cong \mathbb{R}^3$ its Lie algebra. The motion of the rigid body is invariant under $SO(3)$. The reduced Lagrangian function for this system defined on the Lie algebroid $E = \mathfrak{so}(3)$, $\ell : \mathfrak{so}(3) \rightarrow \mathbb{R}$ is given by

$$\ell(\Omega_1, \Omega_2, \Omega_3) = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2).$$

Denote by $t \rightarrow R(t) \in SO(3)$ a curve. The columns of the matrix $R(t)$ represent the directions of the principal axis of the body at time t with respect to some reference system. Consider the following left invariant control problem. First, we have the reconstruction equation:

$$\dot{R}(t) = R(t) \begin{pmatrix} 0 & -\Omega_3(t) & \Omega_2(t) \\ \Omega_3(t) & 0 & -\Omega_1(t) \\ -\Omega_2(t) & \Omega_1(t) & 0 \end{pmatrix} = R(t) (\Omega_1(t) E_1 + \Omega_2(t) E_2 + \Omega_3(t) E_3)$$

where

$$E_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the equations for the angular velocities Ω_i with $i = 1, 2, 3$:

$$I_1 \dot{\Omega}_1(t) = (I_2 - I_3) \Omega_2(t) \Omega_3(t) + u_1(t)$$

$$I_2 \dot{\Omega}_2(t) = (I_3 - I_1) \Omega_3(t) \Omega_1(t) + u_2(t)$$

$$I_3 \dot{\Omega}_3(t) = (I_1 - I_2) \Omega_1(t) \Omega_2(t) + u_3(t)$$

where I_1, I_2, I_3 are the moments of inertia and u_1, u_2, u_3 denote the applied torques playing the role of controls of the system.

The optimal control problem for the rigid body consists on finding the trajectories $(R(t), \Omega(t), u(t))$ with fixed initial and final conditions $(R(0), \Omega(0))$, $(R(T), \Omega(T))$ respectively and minimizing the cost functional

$$\mathcal{A} = \int_0^T \mathcal{C}(\Omega, u_1, u_2, u_3) dt = \frac{1}{2} \int_0^T (u_1^2 + u_2^2 + u_3^2) dt.$$

This optimal control problem is equivalent to solve the following second-order (unconstrained) variational problem

$$\min \tilde{\mathcal{J}} = \int_0^T \tilde{L}(\Omega, \dot{\Omega}) dt$$

where

$$\tilde{L}(\Omega, \dot{\Omega}) = \mathcal{C} \left(\Omega, I_1 \dot{\Omega}_1 - (I_2 - I_3) \Omega_2 \Omega_3, I_2 \dot{\Omega}_2 - (I_3 - I_1) \Omega_3 \Omega_1, I_3 \dot{\Omega}_3 - (I_1 - I_2) \Omega_1 \Omega_2 \right),$$

that is,

$$\tilde{L}(\Omega, \dot{\Omega}) = \frac{1}{2} \left[\left(I_1 \dot{\Omega}_1 - (I_2 - I_3) \Omega_2 \Omega_3 \right)^2 + \left(I_2 \dot{\Omega}_2 - (I_3 - I_1) \Omega_3 \Omega_1 \right)^2 + \left(I_3 \dot{\Omega}_3 - (I_1 - I_2) \Omega_1 \Omega_2 \right)^2 \right].$$

Next, for simplicity, we consider the particular case $I_1 = I_2 = I_3 = 1$. The second order Lagrangian is given by

$$\tilde{L}(\Omega, \dot{\Omega}) = \frac{1}{2} \left(\dot{\Omega}_1^2 + \dot{\Omega}_2^2 + \dot{\Omega}_3^2 \right).$$

The Pontryagin bundle is $W_0 = 2\mathfrak{so}(3) \times 2\mathfrak{so}(3)^*$ with induced coordinates

$$(\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_1, \dot{\Omega}_2, \dot{\Omega}_3, p_1, p_2, p_3, \bar{p}_1, \bar{p}_2, \bar{p}_3).$$

The first constraint submanifold is given by

$$W_1 = \{\bar{p}_1 - \dot{\Omega}_1 = 0, \quad \bar{p}_2 - \dot{\Omega}_2 = 0, \quad \bar{p}_3 - \dot{\Omega}_3 = 0\}.$$

Observe that

$$\left(\frac{\partial^2 \tilde{L}}{\partial \dot{\Omega}_A \partial \dot{\Omega}_B} \right) = \mathbf{I}_{3 \times 3}$$

where $\mathbf{I}_{3 \times 3}$ denotes the 3×3 identity matrix. Thus, the constraint algorithm stops at the first constraint submanifold W_1 .

We can write the equations of motion for the optimal control system as:

$$\begin{aligned} \dot{p}_1 &= p_3 \Omega_2 - p_2 \Omega_3, & \frac{d}{dt} \Omega_1 &= \dot{\Omega}_1, \\ \dot{p}_2 &= p_1 \Omega_3 - p_3 \Omega_1, & \frac{d}{dt} \Omega_2 &= \dot{\Omega}_2, \\ \dot{p}_3 &= p_2 \Omega_1 - p_1 \Omega_2, & \frac{d}{dt} \Omega_3 &= \dot{\Omega}_3, \\ \dot{\bar{p}}_1 &= -p_1, & \dot{\Omega}_1 &= \bar{p}_1, \\ \dot{\bar{p}}_2 &= -p_2, & \dot{\Omega}_2 &= \bar{p}_2, \\ \dot{\bar{p}}_3 &= -p_3, & \dot{\Omega}_3 &= \bar{p}_3. \end{aligned}$$

After some strighforward computations, previous equations can be reduced to

$$\ddot{\Omega}_1 = \Omega_3 \ddot{\Omega}_2 - \Omega_2 \ddot{\Omega}_3, \quad \ddot{\Omega}_2 = \Omega_1 \ddot{\Omega}_3 - \Omega_3 \ddot{\Omega}_1, \quad \ddot{\Omega}_3 = \Omega_2 \ddot{\Omega}_1 - \Omega_1 \ddot{\Omega}_2.$$

or in short notation,

$$\ddot{\Omega} = -\Omega \times \ddot{\Omega}.$$

The previous equations are the equations given by L. Noakes, G. Heinzinger and B. Paden, [63] for cubic splines on $SO(3)$.

Finally, we would like to comment that the regularity condition provides the existence of a unique solution of the dynamics along the submanifold W_1 . Therefore, there exists a

unique vector field $X \in \mathfrak{X}(W_1)$ which satisfies $i_X \Omega_{W_1} = dH_{W_1}$. In consequence, we have a unique control input which extremizes (minimizes) the objective function \mathcal{A} . If we take the flow $F_t : W_1 \rightarrow W_1$ of the vector field X then we have that $F_t^* \Omega_{W_1} = \Omega_{W_1}$. Obviously, the Hamiltonian function

$$H_{W_0}(\Omega, \dot{\Omega}, p, \bar{p}) = \bar{p}_A \dot{\Omega}_A + p_A \dot{\Omega}_A - \frac{1}{2} (\dot{\Omega}_1^2 + \dot{\Omega}_2^2 + \dot{\Omega}_3^2)$$

is preserved by the solution of the optimal control problem, that is $\tilde{H}|_{W_1} \circ F_t = \tilde{H}|_{W_1}$.

4.2. Optimal control problems of underactuated mechanical systems on Lie algebroids. Now, suppose that our mechanical control system is underactuated, that is, the number of control inputs is less than the dimension of the configuration space. The class of underactuated mechanical systems is abundant in real life for different reasons; for instance, as a result of design choices motivated by the search of less cost engineering devices or as a result of a failure regime in fully actuated mechanical systems. Underactuated systems include spacecrafts, underwater vehicles, mobile robots, helicopters, wheeled vehicles and underactuated manipulators. In the general situation, the dynamics is specified fixed a Lagrangian $L : E \rightarrow \mathbb{R}$ where $(E, [\cdot, \cdot], \rho)$ is a Lie algebroid over a manifold Q with fiber bundle projection $\tau_E : E \rightarrow Q$.

If we take local coordinates (q^i) on Q and a local basis $\{e_A\}$ of sections of E , then we have the corresponding local coordinates (q^i, y^A) on E . Such coordinates determine the local structure functions ρ_A^i and C_{AB}^C and then the Euler-Lagrange equations on Lie algebroids can be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^A} = 0.$$

These equations are precisely the components of the *Euler-Lagrange operator* $\mathcal{E}L : E^{(2)} \rightarrow E^*$, which locally reads

$$\mathcal{E}L = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^A} \right) e^A,$$

where $\{e^A\}$ is the dual basis of $\{e_A\}$ (see [26]). In terms of the Euler-Lagrange operator, the equations of motion just read $\mathcal{E}L = 0$.

In the underactuated case, we model the set of control forces by the vector subbundle $\text{span}\{e^a\} \subset E^*$ and the forces are given by $u_F = (u_F)_a e^a$.

Now, we add controls in our picture. Assume that the controlled Euler-Lagrange equations are

$$\left(\frac{d}{dt} \left(\frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^A} \right) e^A = u_a e^a, \quad (38)$$

where we are denoting as $\{e^A\} = \{e^a, e^\alpha\}$ the dual basis of $\{e_A\}$ and u_a are admissible control parameters. Using the basis of sections of E , equations (38) can be rewritten as

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial y^a} \right) - \rho_a^i \frac{\partial L}{\partial q^i} + C_{aB}^C y^B \frac{\partial L}{\partial y^C} &= u_a, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) - \rho_\alpha^i \frac{\partial L}{\partial q^i} + C_{\alpha B}^C y^B \frac{\partial L}{\partial y^C} &= 0. \end{aligned} \quad (39)$$

The optimal control problem consists on finding an admissible curve $\gamma(t) = (q^i(t), y^A(t), u(t))$ of the state variables and control inputs given initial and final boundary conditions $(q^i(0), y^A(0))$ and $(q^i(T), y^A(T))$, respectively, solving the controlled Euler-Lagrange equations (39) and minimizing

$$\mathcal{A}(q^i, y^A, u_a) = \int_0^T C(q^i, y^A, u_a) dt,$$

where $C : E \times U \rightarrow \mathbb{R}$ denotes the cost function.

To solve this optimal control problem is equivalent to solve the following second-order problem:

$$\begin{aligned} \min \quad & \tilde{L}(q^i(t), y^A(t), z^A(t)) \\ \text{subject to } & \Phi^\alpha(q^i(t), y^A(t), z^A(t)), \alpha = 1, \dots, m \end{aligned}$$

where $\tilde{L}, \Phi^\alpha \in C^\infty(E^{(2)})$. Here

$$\tilde{L}(q^i(t), y^A(t), z^A(t)) = C\left(q^i(t), y^A(t), F_a(x^i(t), y^A(t), z^A(t))\right),$$

where

$$F_a(q^i(t), y^A(t), z^A(t)) = \frac{d}{dt} \left(\frac{\partial L}{\partial y^a} \right) - \rho_a^i \frac{\partial L}{\partial q^i} + C_{aB}^C y^B \frac{\partial L}{\partial y^C}.$$

The Lagrangian \tilde{L} is subjected to the second-order constraints:

$$\Phi^\alpha(q^i(t), y^A(t), z^A(t)) = \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) - \rho_\alpha^i \frac{\partial L}{\partial q^i} + C_{\alpha B}^C y^B \frac{\partial L}{\partial y^C},$$

which determines a submanifold \mathcal{M} of $E^{(2)}$.

Remark 4. Observe that the cost function is not completely defined in $E \oplus E^*$, it is only defined in a smaller subset of this space because $\frac{d}{dt} \left(\frac{\partial L}{\partial y^a} \right) - \rho_a^i \frac{\partial L}{\partial q^i} + C_{aB}^C y^B \frac{\partial L}{\partial y^C}$ only belongs to the vector subbundle $\text{span}\{e^a\} \subset E^*$. That is, in the case of fully actuated system the cost function would be defined in the full space E^* , and when we are dealing with an underactuated systems, the cost function is defined in a proper subset of E^* . Next, for simplicity, we assume that $C : E \oplus E^* \rightarrow \mathbb{R}$. \diamond

Observe that from the constraint equations we have that

$$\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} z^\beta + \frac{\partial^2 L}{\partial y^\alpha \partial y^a} z^a - \rho_\alpha^i \frac{\partial L}{\partial q^i} + C_{\alpha B}^C y^B \frac{\partial L}{\partial y^C} = 0.$$

Therefore, assuming that the matrix $W_{\alpha\beta} = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \right)$ is regular, we can write the equations as

$$z^\alpha = -W^{\alpha\beta} \left(\frac{\partial^2 L}{\partial y^\beta \partial y^a} z^a - \rho_\beta^i \frac{\partial L}{\partial q^i} + C_{\beta B}^C y^B \frac{\partial L}{\partial y^C} \right) = G^\alpha(q^i, y^A, z^a)$$

where $W^{\alpha\beta} = (W_{\alpha\beta})^{-1}$.

Therefore, we can choose coordinates (q^i, y^A, z^a) on \mathcal{M} . This choice allows us to consider an intrinsic point of view, that is, to work directly on $\bar{W} = \mathcal{M} \times (\mathcal{T}^{\tau_E} E)^*$ avoiding the use of the Lagrange multipliers.

Define the restricted Lagrangian $\tilde{L}_\mathcal{M}$ by $\tilde{L}|_\mathcal{M} : \mathcal{M} \rightarrow \mathbb{R}$ and take induced coordinates on \bar{W} , $(q^i, y^A, z^a, p_A, \bar{p}_A)$. Applying the same procedure than in section 3.4.3 we derive the following system of equations

$$\begin{aligned} \dot{q}^i &= \rho_A^i y^A, \\ \frac{dy^a}{dt} &= z^a, \\ \frac{dy^\alpha}{dt} &= G^\alpha(q^i, y^A, z^a), \\ \frac{dp_A}{dt} &= \rho_A^i \left(\frac{\partial \tilde{L}_\mathcal{M}}{\partial q^i} - \bar{p}_\beta \frac{\partial G^\beta}{\partial q^i} \right) + C_{AB}^C p_C y^B, \\ \frac{d\bar{p}_A}{dt} &= -p_A + \frac{\partial \tilde{L}_\mathcal{M}}{\partial y^A} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^A}, \\ \bar{p}_a &= \frac{\partial \tilde{L}_\mathcal{M}}{\partial z^a} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^a}. \end{aligned}$$

To shorten the number of unknown variables involved in the previous set of equations, we can write them using as variables $(q^i, y^A, z^a, \bar{p}_\alpha)$

$$\begin{aligned}
\dot{q}^i &= \rho_A^i y^A, \\
\frac{dy^\alpha}{dt} &= G^\alpha(q^i, y^A, z^a), \\
0 &= \frac{d^2}{dt^2} \left(\frac{\partial \tilde{L}_M}{\partial z^a} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^a} \right) - C_{Aa}^b y^A \left(\frac{d}{dt} \left[\frac{\partial \tilde{L}_M}{\partial z^b} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^b} \right] \right) - \frac{d}{dt} \left(\frac{\partial \tilde{L}_M}{\partial y^a} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^a} \right) \\
&\quad + C_{Aa}^C y^A \left(\frac{\partial \tilde{L}_M}{\partial y^C} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^C} \right) + \rho_a^i \left(\frac{\partial \tilde{L}_M}{\partial q^i} - \bar{p}_\beta \frac{\partial G^\beta}{\partial q^i} \right) - C_{Aa}^\gamma y^A \frac{d\bar{p}_\gamma}{dt}, \\
0 &= \frac{d^2 \bar{p}_\alpha}{dt^2} + C_{A\alpha}^\beta y^A \frac{d\bar{p}_\beta}{dt} - C_{A\alpha}^C y^A \left[\frac{\partial \tilde{L}_M}{\partial y^C} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^C} \right] - \frac{d}{dt} \left[\frac{\partial \tilde{L}_M}{\partial y^\alpha} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^\alpha} \right] \\
&\quad + \rho_\alpha^i \left(\frac{\partial \tilde{L}_M}{\partial q^i} - \bar{p}_\beta \frac{\partial G^\beta}{\partial q^i} \right) + C_{A\alpha}^b y^A \left(\frac{d}{dt} \left[\frac{\partial \tilde{L}_M}{\partial z^b} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^b} \right] \right) - C_{A\alpha}^b y^A \left[\frac{\partial \tilde{L}_M}{\partial y^b} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^b} \right]
\end{aligned}$$

If the matrix

$$\left(\frac{\partial^2 \tilde{L}_M}{\partial z^a \partial z^b} \right)$$

is regular then we can write the previous equations as an explicit system of third-order differential equations. This regularity assumption is equivalent to the condition that the constrain algorithm stops at the first constraint submanifold. In this submanifold there exists a unique solution for the boundary value problem determined by the optimal control problem.

Example 28. Optimal control of an underactuated Elroy's beanie: This mechanical system is probably the simplest example of a dynamical system with a non-Abelian Lie group symmetry. It consists of two planar rigid bodies connected through their centers of mass (by a rotor let's say) moving freely in the plane (see [5] and [64]). The main (i.e. more massive) rigid body has the capacity to apply a torque to the connected rigid body.

The configuration space is $Q = SE(2) \times S^1$ with coordinates (x, y, θ, ψ) , where the first three coordinates describe the position and orientation of the center of mass of the first body and the last one describe the relative orientation between both bodies.

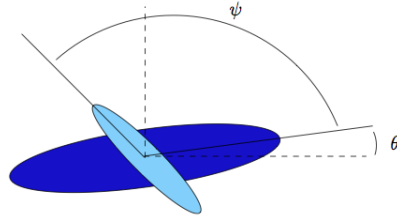


FIGURE 2. Top View of Elroy's beanie.

The Lagrangian $L : TQ \rightarrow \mathbb{R}$ is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2 - V(\psi)$$

where m denotes the mass of the system and I_1 and I_2 are the inertias of the first and the second body, respectively; additionally, we also consider a potential function of the form $V(\psi)$. The kinetic energy is associated with the Riemannian metric \mathcal{G} on Q given by

$$\mathcal{G} = m(dx^2 + dy^2) + (I_1 + I_2)d\theta^2 + I_2d\theta \otimes d\psi + I_2d\psi \otimes d\theta + I_2d\psi^2.$$

The system is $SE(2)$ -invariant for the action

$$\Phi_g(q) = (z_1 + x \cos \alpha - y \sin \alpha, z_2 + x \sin \alpha + y \cos \alpha, \alpha + \theta, \psi)$$

where $g = (z_1, z_2, \alpha)$.

Let $\{\xi_1, \xi_2, \xi_3\}$ be the standard basis of $\mathfrak{se}(2)$,

$$[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = -\xi_2, \quad [\xi_2, \xi_3] = \xi_1.$$

The quotient space $\widehat{Q} = Q/SE(2) = (SE(2) \times S^1)/SE(2) \simeq S^1$ is naturally parameterized by the coordinate ψ . The Atiyah algebroid $TQ/SE(2) \rightarrow \widehat{Q}$ is identified with the vector bundle: $\tau_{\bar{A}} : \bar{A} = TS^1 \times \mathfrak{se}(2) \rightarrow S^1$. The canonical basis of sections of $\tau_{\bar{A}}$ is: $\left\{ \frac{\partial}{\partial \psi}, \xi_1, \xi_2, \xi_3 \right\}$. Since the metric \mathcal{G} is also $SE(2)$ -invariant we obtain a bundle metric $\hat{\mathcal{G}}$ and a $\hat{\mathcal{G}}$ -orthonormal basis of sections:

$$\left\{ X_1 = \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \left(\frac{\partial}{\partial \psi} - \frac{I_2}{I_1 + I_2} \xi_3 \right), X_2 = \frac{1}{\sqrt{m}} \xi_1, X_3 = \frac{1}{\sqrt{m}} \xi_2, X_4 = \frac{1}{\sqrt{I_1 + I_2}} \xi_3 \right\}$$

In the coordinates $(\psi, v^1, v^2, v^3, v^4)$ induced by the orthonormal basis of sections, the reduced Lagrangian is

$$\bar{L} = \frac{1}{2} ((v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2) - V(\psi).$$

Additionally, we deduce that

$$\begin{aligned} \llbracket X_1, X_2 \rrbracket_{\bar{A}} &= -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}} X_3, & \llbracket X_1, X_3 \rrbracket_{\bar{A}} &= \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} X_2, \\ \llbracket X_1, X_4 \rrbracket_{\bar{A}} &= 0, & \llbracket X_2, X_3 \rrbracket_{\bar{A}} &= 0, \\ \llbracket X_2, X_4 \rrbracket_{\bar{A}} &= -\frac{1}{\sqrt{I_1 + I_2}} X_3, & \llbracket X_3, X_4 \rrbracket_{\bar{A}} &= \frac{1}{\sqrt{I_1 + I_2}} X_2. \end{aligned}$$

Therefore, the non-vanishing structure functions are

$$C_{12}^3 = -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}}, \quad C_{13}^2 = \sqrt{\frac{I_2}{I_1(I_1 + I_2)}}, \quad C_{24}^3 = -\frac{1}{\sqrt{I_1 + I_2}}, \quad C_{34}^2 = \frac{1}{\sqrt{I_1 + I_2}}.$$

Moreover,

$$\rho_{\bar{A}}(X_1) = \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial}{\partial \psi}, \quad \rho_{\bar{A}}(X_2) = 0, \quad \rho_{\bar{A}}(X_3) = 0, \quad \rho_{\bar{A}}(X_4) = 0.$$

The local expression of the Euler-Lagrange equations for the reduced Lagrangian system $\bar{L} : \bar{A} \rightarrow \mathbb{R}$ is:

$$\begin{aligned}\dot{\psi} &= \sqrt{\frac{I_1 + I_2}{I_1 I_2}} v^1, \\ \dot{v}^1 &= -\sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi}, \\ \dot{v}^2 &= -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^3 + \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\ \dot{v}^3 &= \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^2 - \frac{1}{\sqrt{I_1 + I_2}} v^2 v^4, \\ \dot{v}^4 &= 0.\end{aligned}$$

Next we introduce controls in our picture. Let $u(t) \in \mathbb{R}$ be a control input that permits to steer the system from an initial position to a desired position by controlling only the variable ψ . Therefore the controlled Euler-Lagrange equations are now

$$\begin{aligned}\dot{\psi} &= \sqrt{\frac{I_1 + I_2}{I_1 I_2}} v^1, \\ \dot{v}^1 &= -\sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi} + u, \\ \dot{v}^2 &= -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^3 + \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\ \dot{v}^3 &= \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^2 - \frac{1}{\sqrt{I_1 + I_2}} v^2 v^4, \\ \dot{v}^4 &= 0.\end{aligned}$$

From the second equation we obtain the feedback control law:

$$u = \dot{v}^1 + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi}.$$

The optimal control problem consists of finding trajectories of the states variables and controls inputs, satisfying the previous equations subject to given initial and final conditions and minimizing the cost functional,

$$\min_{(v, \psi, \dot{\psi}, u)} \int_0^T C(v, \psi, \dot{\psi}, u) dt = \min_{(\psi, \dot{\psi}, \Omega, u)} \int_0^T \frac{1}{2} u^2 dt$$

where $v = (v^1, v^2, v^3, v^4)$.

Our optimal control problem is equivalent to solving the following second-order variational problem with second-order constraints given by

$$\min_{(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi})} \widetilde{L}(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) = C \left(v, \psi, \dot{\psi}, \dot{v}^1 + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi} \right),$$

where $\widetilde{L} : T^{(2)}\mathbb{S}^1 \times \widetilde{2SE(2)} \rightarrow \mathbb{R}$, subject the second-order constraints $\Phi^\alpha : T^{(2)}\mathbb{S}^1 \times \widetilde{2SE(2)} \rightarrow \mathbb{R}$, $\alpha = 1, \dots, 4$,

$$\begin{aligned}
\Phi^1(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{\psi} - \sqrt{\frac{I_2 + I_1}{I_2 I_1}} v^1, \\
\Phi^2(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{v}^2 - \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4 + \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^3, \\
\Phi^3(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{v}^3 + \frac{1}{\sqrt{I_1 + I_2}} v^2 v^4 - \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^2, \\
\Phi^4(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{v}^4.
\end{aligned}$$

Therefore, the constraint submanifold \mathcal{M} of $T^{(2)}\mathbb{S}^1 \times \widetilde{2SE(2)}$ is given by

$$\begin{aligned}
\mathcal{M} = \left\{ (v, \dot{v}, \psi, \dot{\psi}) \mid \dot{\psi} = \sqrt{\frac{I_2 + I_1}{I_2 I_1}} v^1, \dot{v}^2 = \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4 - \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^3, \right. \\
\left. \dot{v}^3 = -\frac{1}{\sqrt{I_1 + I_2}} v^2 v^4 + \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^2, \dot{v}^4 = 0 \right\}.
\end{aligned}$$

We consider the submanifold $W_0 = \mathcal{M} \times \widetilde{2SE(2)}^*$ with induced coordinates

$$(v^1, v^2, v^3, v^4, \psi, \dot{v}^1, p_1, p_2, p_3, p_4, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4).$$

Now, we consider the restriction $\tilde{L}_{\mathcal{M}}$ given by

$$\tilde{L}_{\mathcal{M}} = \frac{1}{2} \left(\dot{v}^1 + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi} \right)^2.$$

Moreover, the first constraint submanifold W_1 is determined by

$$W_1 = \left\{ z \in W_0 \mid \bar{p}_1 - \dot{v}^1 - \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \left(\frac{\partial V}{\partial \psi} + \bar{p}_1 \right) - \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} (\bar{p}_3 v^2 - \bar{p}_2 v^3) = 0 \right\}.$$

Observe that

$$\det \left(\frac{\partial^2 \tilde{L}_{\mathcal{M}}}{\partial \dot{v}^1 \partial \dot{v}^1} \right) \neq 0.$$

Thus, the constraint algorithm stops at the first constraint submanifold W_1 .

Finally, in a similar fashion as the unconstrained situation, we would like to point out that the regularity condition provides the existence of a unique solution of the dynamics along the submanifold W_1 .

Then, we can write the equations determining necessary conditions for the optimal control problem:

$$\begin{aligned}
\dot{p}_1 &= \frac{I_1 + I_2}{I_1 I_2} \left(\dot{v}_1 + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi} \right) \frac{\partial^2 V}{\partial \psi \partial \psi} - \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} p_3 v^2, \\
\dot{p}_2 &= -\frac{p_3 v^4}{\sqrt{I_1 + I_2}} + \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} p_3 v^1, \\
\dot{p}_3 &= \frac{p_2 v^4}{\sqrt{I_1 + I_2}} - \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} p_2 v^1, \\
\dot{p}_4 &= \frac{1}{\sqrt{I_1 + I_2}} (p_3 v^2 - p_2 v^3), \\
\dot{\bar{p}}_1 &= -p_1 + \bar{p}_1 \sqrt{\frac{I_1 + I_2}{I_1 I_2}} + \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} (\bar{p}_3 v^2 - \bar{p}_2 v^3), \\
\dot{\bar{p}}_2 &= -p_2 + \bar{p}_3 \left(\frac{v_4}{\sqrt{I_1 + I_2}} + v^1 \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} \right), \\
\dot{\bar{p}}_3 &= -p_3 + \bar{p}_2 \left(-\frac{v_4}{\sqrt{I_1 + I_2}} + v^1 \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} \right), \\
\dot{\bar{p}}_4 &= -p_4 + \frac{1}{\sqrt{I_1 + I_2}} (\bar{p}_2 v^3 - \bar{p}_3 v^2), \\
\bar{p}_1 &= \dot{v}^1 + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \left(\frac{\partial V}{\partial \psi} + \bar{p}_1 \right) + \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} (\bar{p}_3 v^2 - \bar{p}_2 v^3), \\
\dot{\psi} &= \sqrt{\frac{I_2 + I_1}{I_2 I_1}} v^1, \quad \dot{v}^2 = \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4 - \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^3, \\
\dot{v}^3 &= -\frac{1}{\sqrt{I_1 + I_2}} v^2 v^4 + \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} v^1 v^2, \quad \dot{v}^4 = 0.
\end{aligned}$$

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